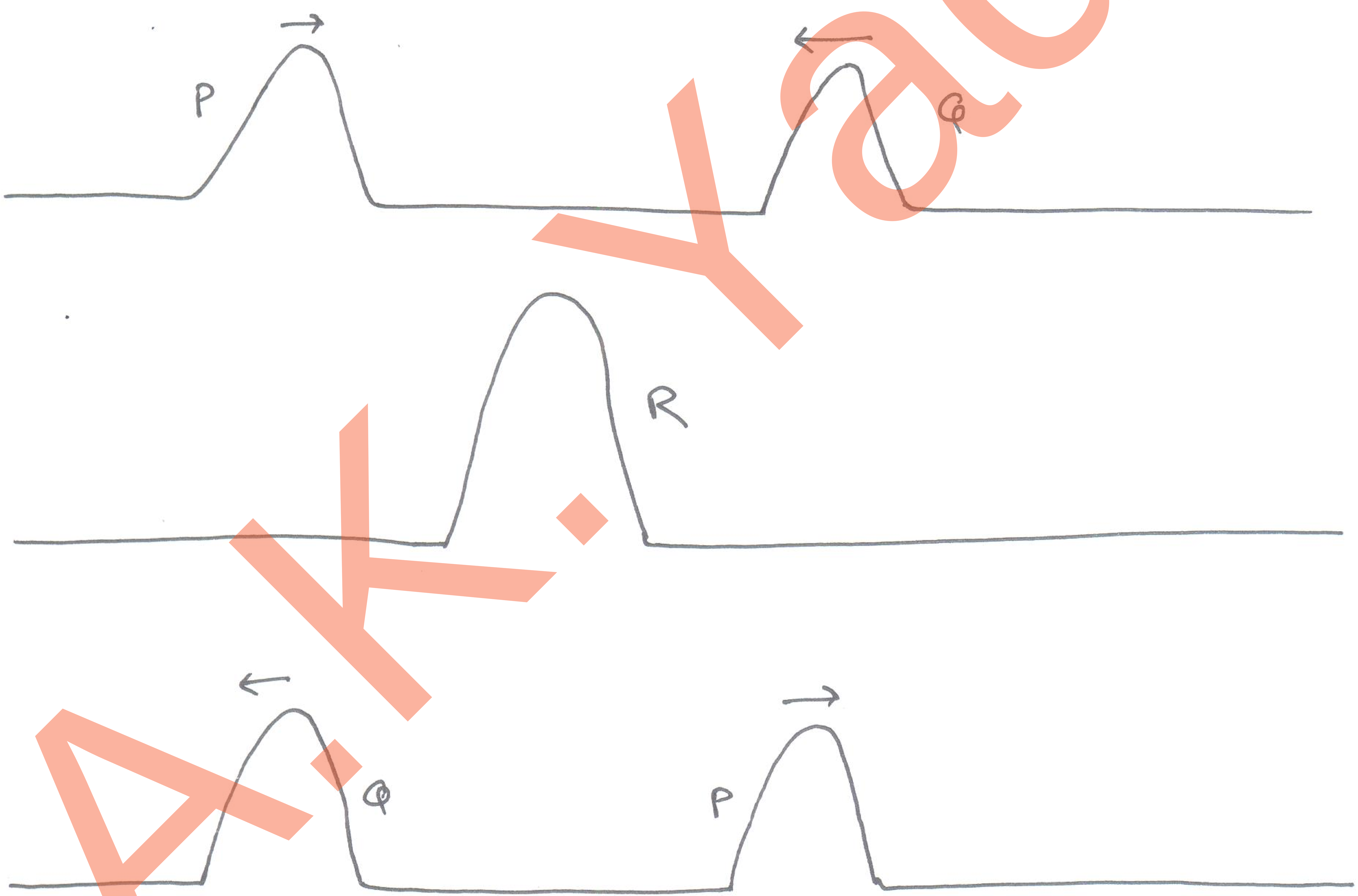
Superposition Principle

When 2 or more waves travel in a medium in such a way that each wave represents its separate motion individually, then the resultant displacement of particle of the medium at any time is equal to the vector sum of the individual displacements.

Consider 'n' waves travelling in a medium superpose on each other.

The resultant displacement is

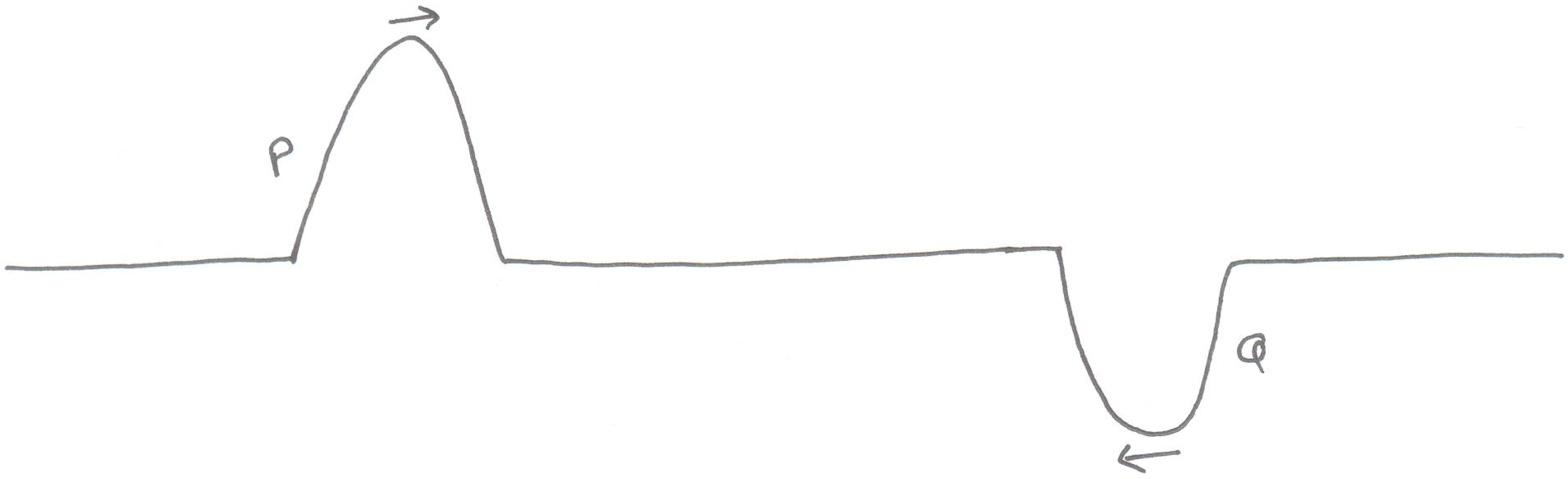
$$\vec{y} = \vec{y}_1 + \vec{y}_2 + \text{------} + \vec{y}_n$$

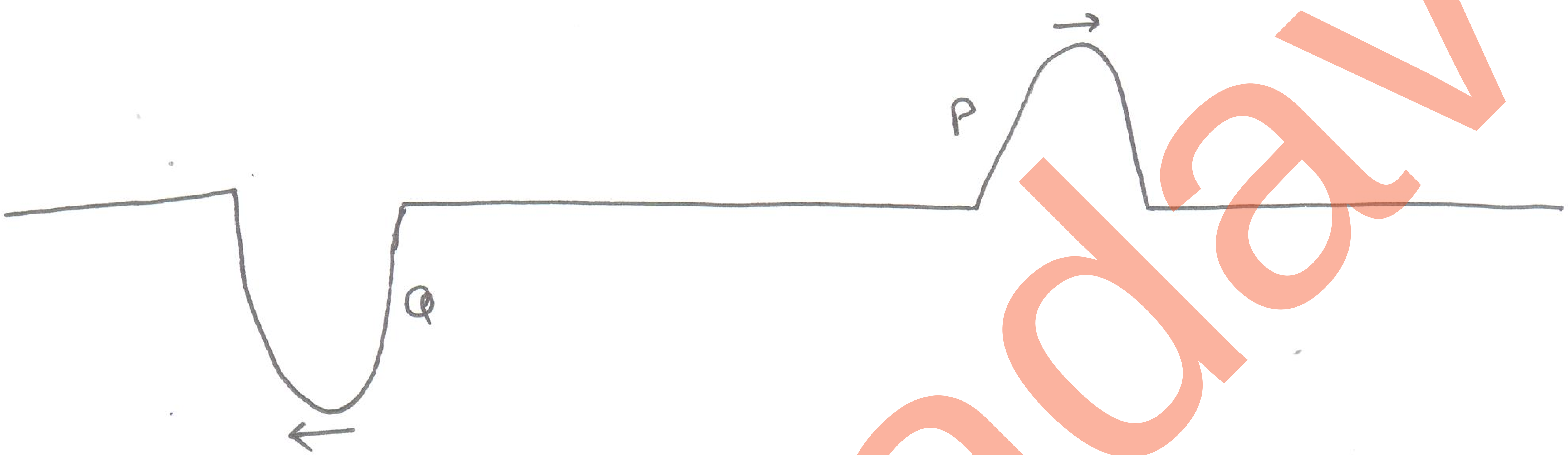


$$\vec{y} = y_1 + y_2$$

$$= \underset{(P)}{A} + \underset{(Q)}{A}$$

$$= 2A\ (R)$$





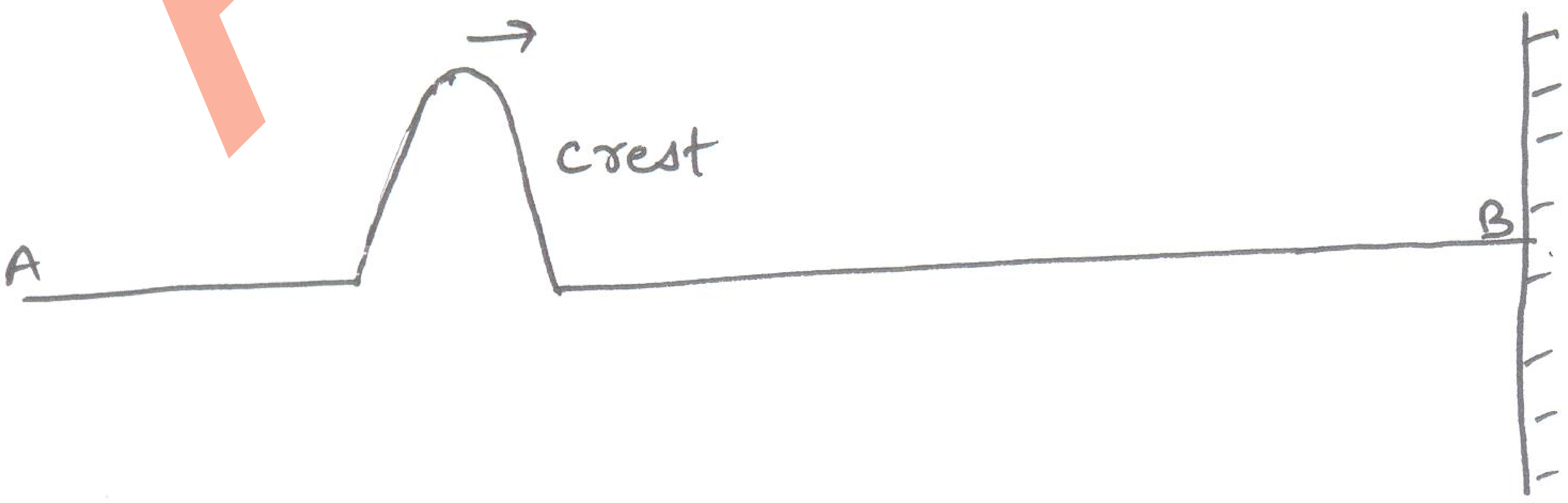
$$y = y_1 + y_2 = \underset{(P)}{A} + \underset{(Q)}{(-A)} = 0 \ (R)$$

* When 2 waves of same frequency moving with same speed & in same direction superpose on each other, they give rise to interference of waves.

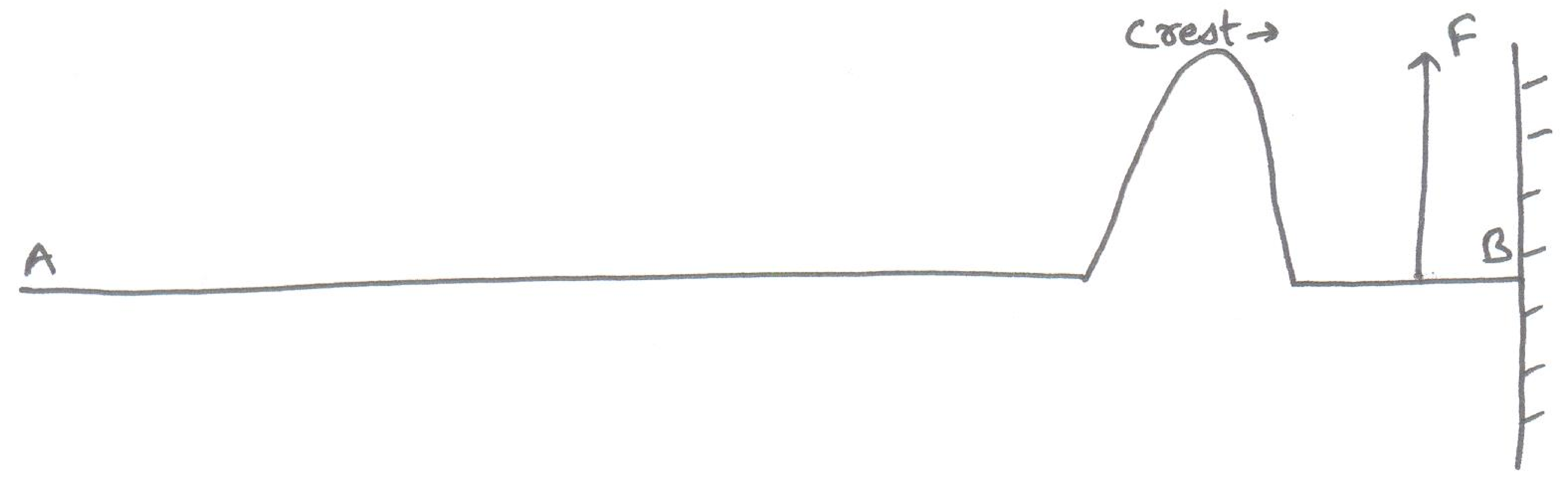
* same speed, same frequency but opp. direction – stationary waves

* same speed, slightly diff. frequencies, same direction – beats.

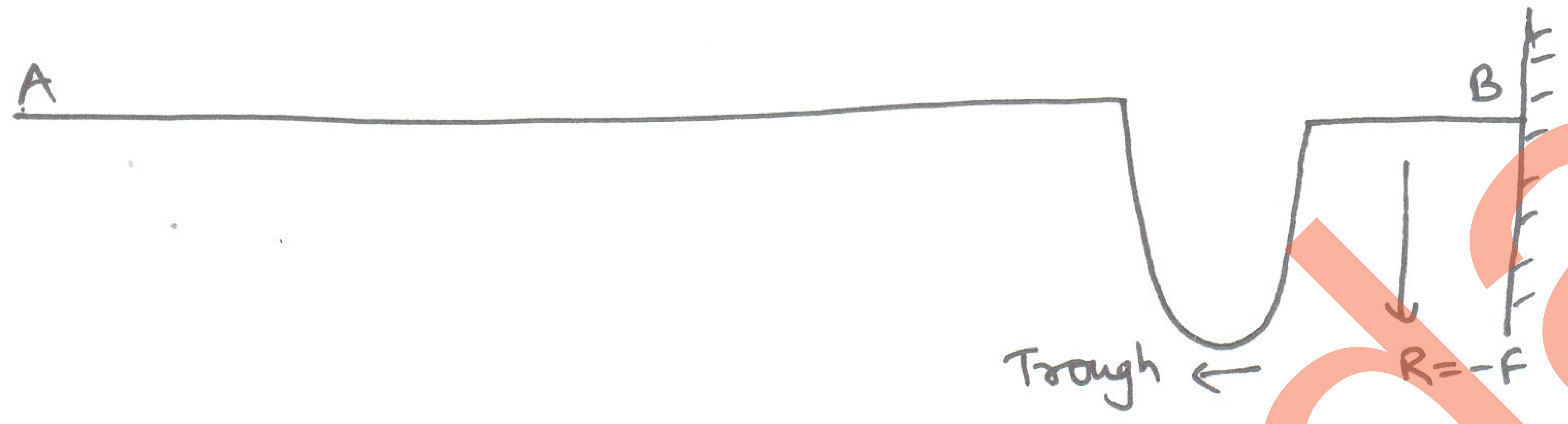
Reflection of transverse waves



Pulse travelling in the form of crest from left to right.



On reaching B, pulse exerts an upward force F on the support. The rigid support remains unaffected & in turn gives an equal & opposite reaction.



A phase change of π takes place in the displacement when a transverse wave is reflected at the rigid support.

Initially, $y = r \sin \frac{2\pi}{\lambda}(vt - x)$

After reflection at rigid support

$$y = r \sin\left[\frac{2\pi}{\lambda}(vt - x) + \pi\right]$$

$$= -r \sin \frac{2\pi}{\lambda}(vt + x)$$

Standing waves in a string fixed at both ends

- Consider a string stretched bet^n 2 fixed points.
- Suppose that a harmonic wave is set up in the stretched string.
- The harmonic wave gets reflected from the 2 fixed ends of the string continuously and a large no. of waves with different phase travelling in either direction are produced in the string.
- These waves interfere with each other.

Consider a wave pulse is moving from left to right along X-axis

$$y' = r \sin \frac{2\pi}{\lambda}(vt - x)$$

Let the right rigid fixed end of the string be origin.

So,
$$y' = r \sin \frac{2\pi}{\lambda}(vt + x) \quad \text{——} (1)$$

After reflection at the right fixed end, the wave pulse travels from right to left.

So,
$$y'' = -r \sin \frac{2\pi}{\lambda}(vt - x)$$

The two waves superimpose on each other and the resultant displacement is given by

$$y = y' + y''$$

$$= r \sin \frac{2\pi}{\lambda}(vt + x) - r \sin \frac{2\pi}{\lambda}(vt - x)$$

$$= r\left[\sin \frac{2\pi}{\lambda}(vt + x) - \sin \frac{2\pi}{\lambda}(vt - x)\right]$$

$$= r \sin \frac{2\pi}{\lambda} x \cos \frac{2\pi}{\lambda} vt \quad \text{——} (2)$$

$$\left[\because \sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}\right]$$

The amplitude of the resultant wave formed is

$$A = 2r \sin \frac{2\pi}{\lambda} x$$

As eqⁿ (2) is not in the form of eqⁿ (1), it does not represent a travelling harmonic wave. It represents a <u>standing wave</u>.

As the string is fixed at both the ends $x=0$ & $x=L$, the ends must be nodes (pt. of min. displacement)

i.e. $y = 0$ for $x = 0$ for all t

$y = 0$ $x = L$ " " "

Eqn 2 satisfies the 1st condition.
It will satisfy 2nd condition if

$$A = 0$$

$$2r \sin \frac{2\pi}{\lambda} L = 0$$

$$\sin \frac{2\pi}{\lambda} L = 0$$

$$\frac{2\pi}{\lambda} L = n\pi$$

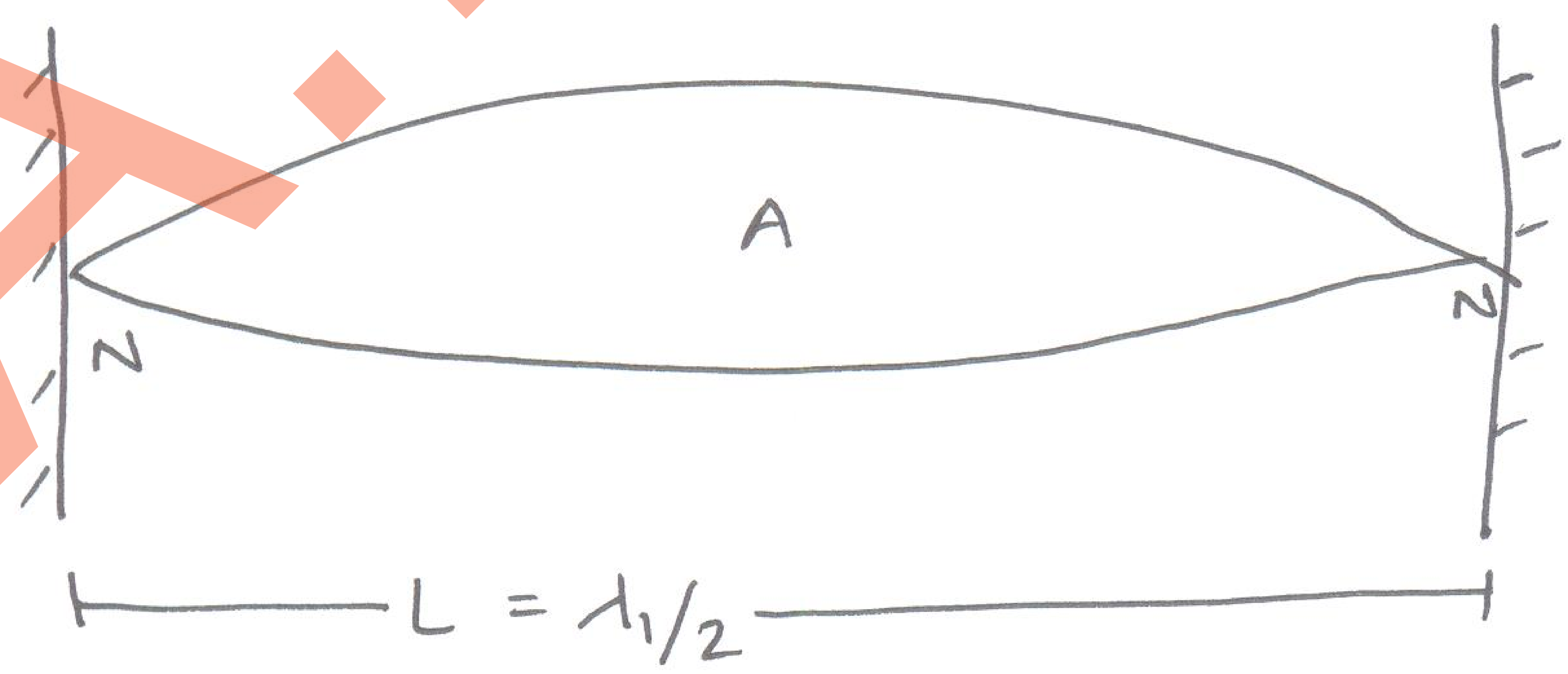
$$L = \frac{n\lambda}{2}$$

<u>first mode of vibration</u>

$$n = 1$$

$$L = \frac{\lambda_1}{2}$$

$$\lambda_1 = 2L$$



$$\nu_1 = \frac{v}{\lambda_1} = \frac{v}{2L}$$

↳ fundamental frequency
Or
Pitch of tone or 1st harmonic

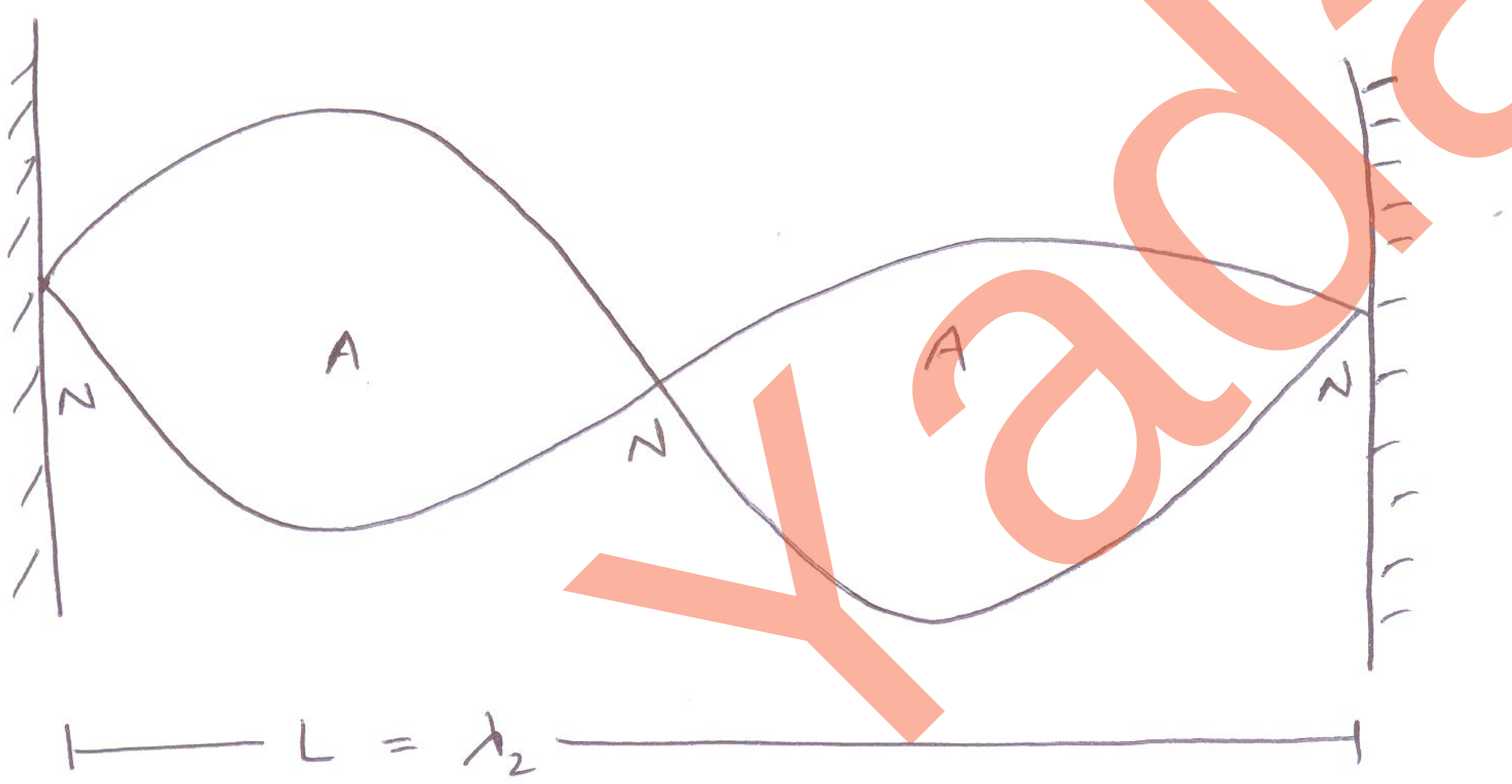
$$v = \sqrt{\frac{T}{m}}$$

$$\therefore \nu_1 = \frac{1}{2L}\sqrt{\frac{T}{m}}$$

Second mode of vibration $\quad n = 2$

$$L = \frac{2\lambda_2}{2} \quad , \quad \lambda_2 = L$$

The string vibrates in 2 segments



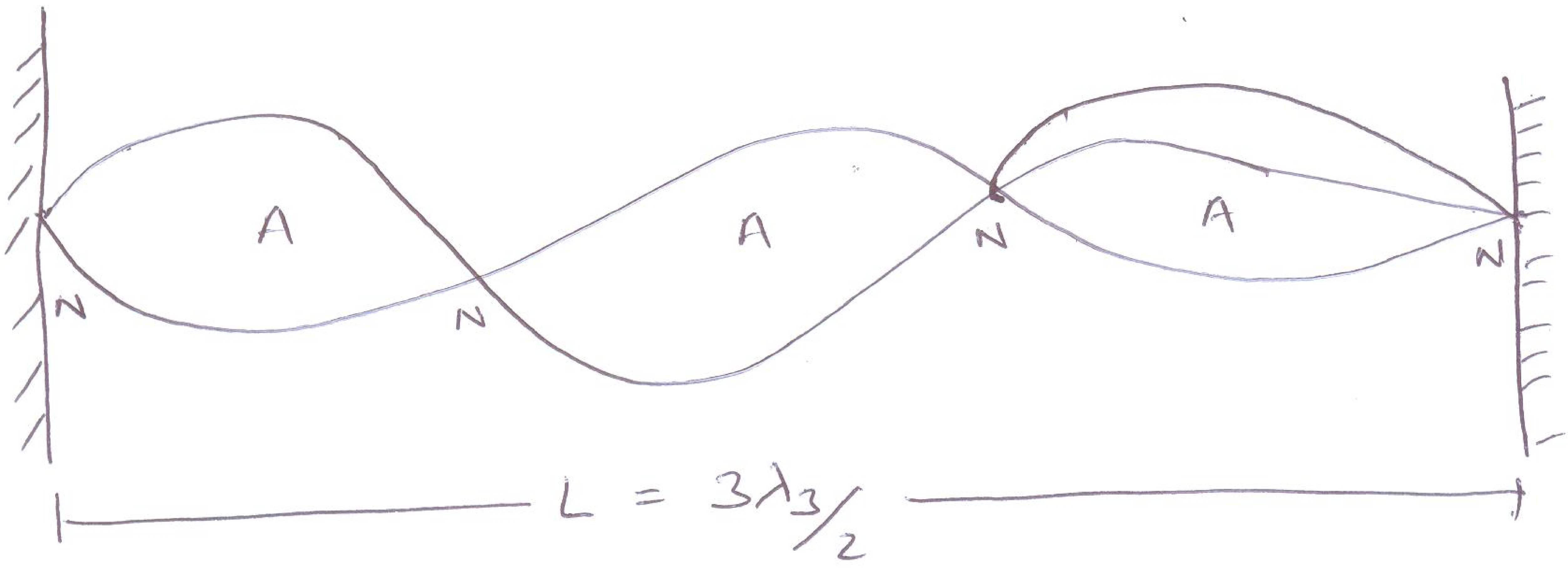
$$\nu_2 = \frac{v}{\lambda_2} = \frac{v}{L} = 2\nu_1$$

↳ 1st overtone or 2nd harmonic

Third mode of vibration $\quad n = 3$

$$L = \frac{3\lambda_3}{2} \quad , \quad \lambda_3 = 2L/3$$

The string will vibrate in 3 segments



$$\nu_3 = \frac{v}{\lambda_3} = \frac{3v}{2L} = 3\nu_1$$

$\searrow$ 2^{nd} overtone or 3^{rd} harmonic

In general, $\lambda_n = \frac{2L}{n}$

$$\nu_n = n\nu_1$$

Nodes – When the standing waves are set up in a string, the points where amplitude is zero are called nodes.

In n^{th} mode of vibration — $(n+1)$ nodes located from right end of the string at distances

$$x = 0, \frac{L}{n}, \frac{2L}{n}, \text{----}, L$$

Anti-nodes – When the standing waves are set up in a string, the points where amplitude is maximum are called anti-nodes.

n^{th} mode — n antinodes located from origin at distances

$$x = \frac{L}{2n}, \frac{3L}{2n}, \text{------} \frac{(2n-1)L}{2n}$$

Standing waves in a pipe closed at one end

Consider a pipe open at the left end & closed at the right end (supposed as origin).

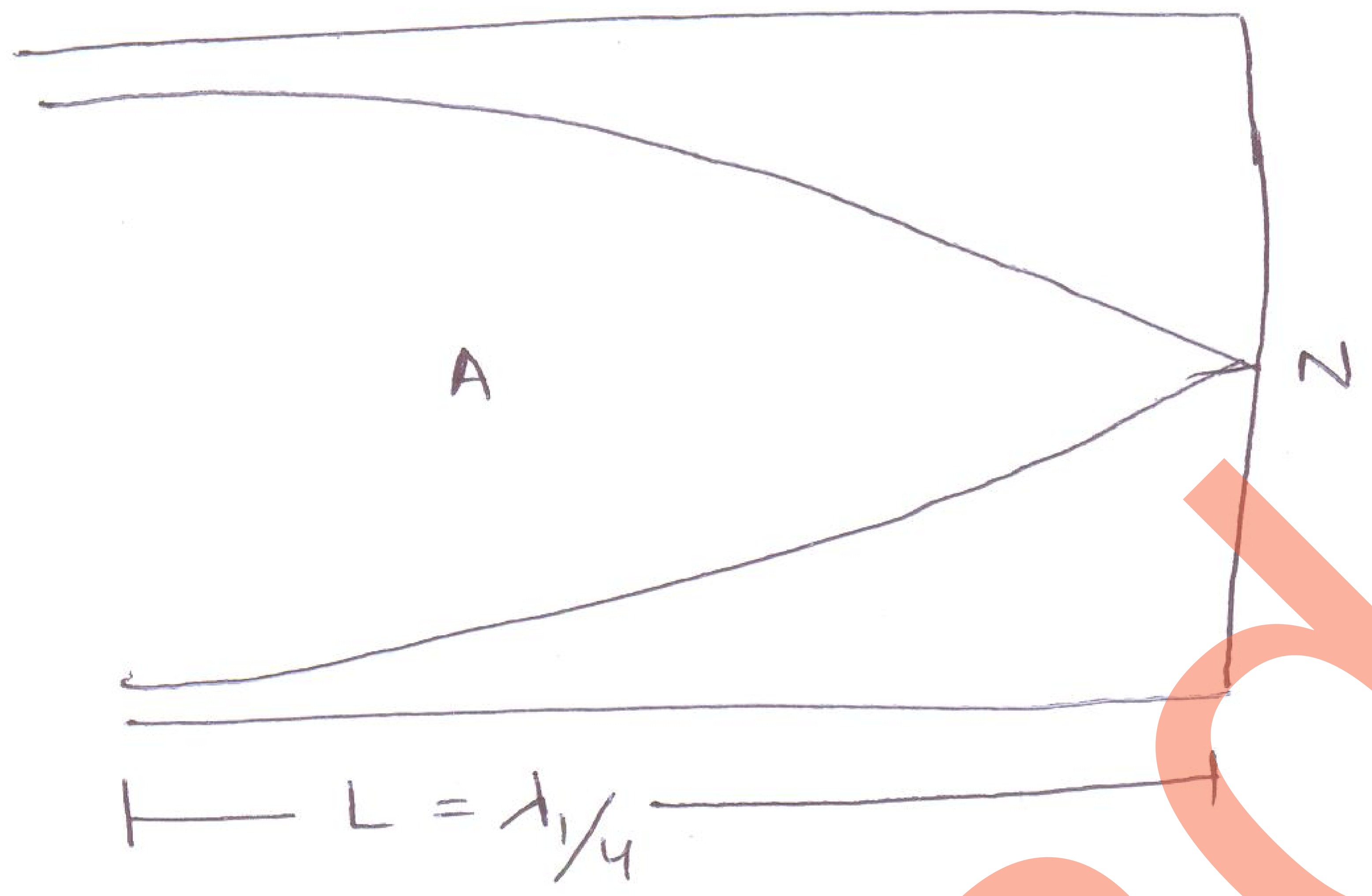
$$y' = r \sin \frac{2\pi}{\lambda}(vt + x)$$

1st mode of vibration $n = 1$

$$L = \lambda_1/4$$

$$\lambda_1 = 4L$$

The length of pipe will be equal to one quarter of the wave length of the waves.

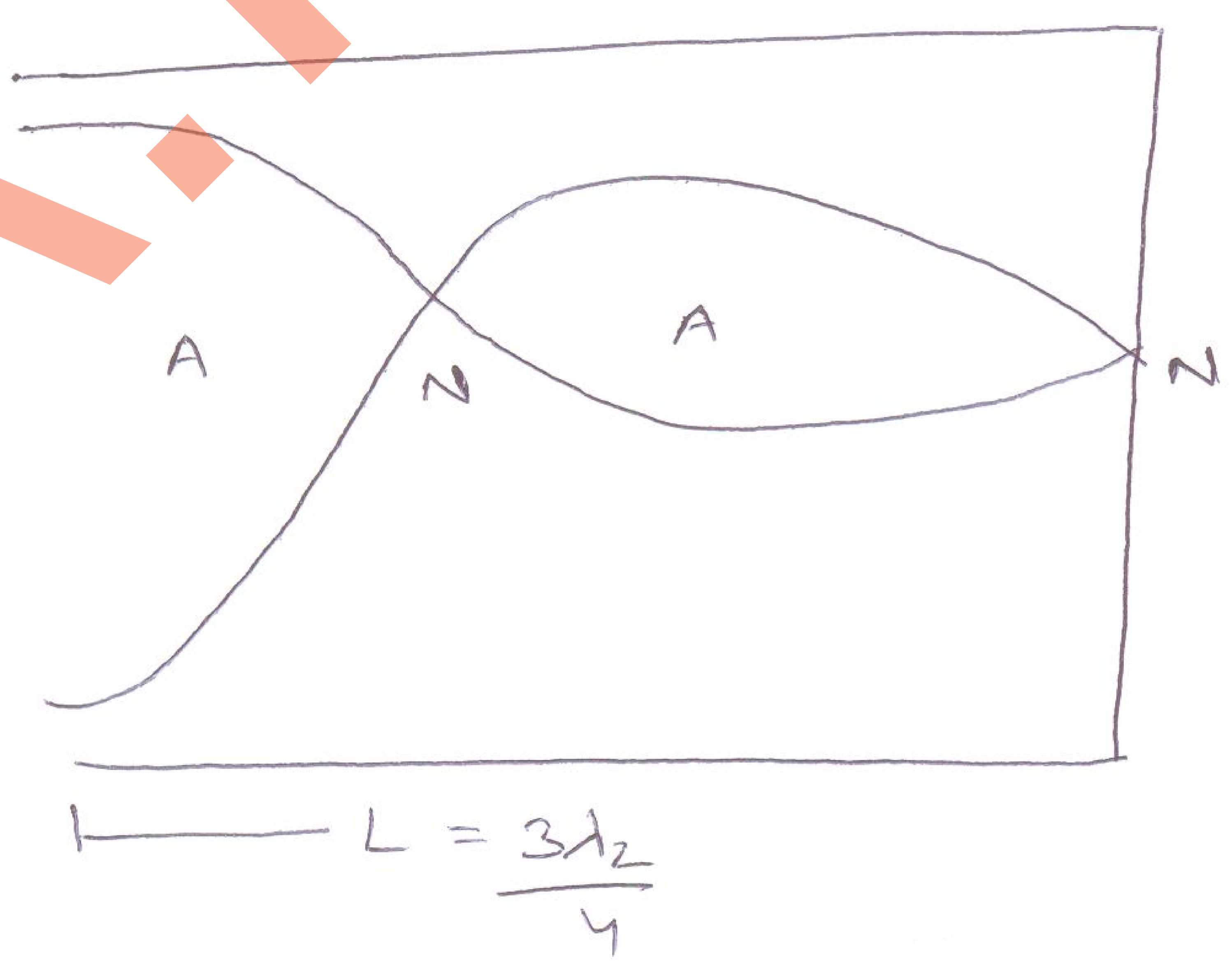


$$\nu_1 = \frac{v}{\lambda_1} = \frac{v}{4L}$$

fundamental frequency or pitch of tone or 1st harmonic

2nd mode of vibration $n = 2$

$$L = \frac{3\lambda_2}{4} \quad \text{or} \quad \lambda_2 = \frac{4L}{3}$$



$$\nu_2 = \frac{v}{\lambda_2} = \frac{3v}{4L} = 3\nu_1$$

1st overtone or 3rd harmonic

When the direct wave pulse meets the rigid boundary at right end, it is reflected back as wave pulse given by

$$y'' = -r \sin \frac{2\pi}{\lambda}(vt - x)$$

The displacement of the resultant wave pulse is

$$y = y' + y''$$

$$= 2r \sin \frac{2\pi}{\lambda} x \cos \frac{2\pi}{\lambda} vt \quad \text{--- (1)}$$

Amplitude, $A = 2r \sin \frac{2\pi}{\lambda} x$

Since the pipe is closed at right end ($x = 0$) & open at left end ($x = L$)

node will be formed at right end

anti-node " " " left "

So, eqn (1) must satisfy the boundary conditions

$y = 0$ at $x = 0$ for all t

$= \text{max.}$ " $x = L$ " "

eqn (1) satisfies the 1st condition.

It will satisfy the 2nd condition if

$$A = \pm 1$$

$$2r \sin \frac{2\pi L}{\lambda} = \pm 1$$

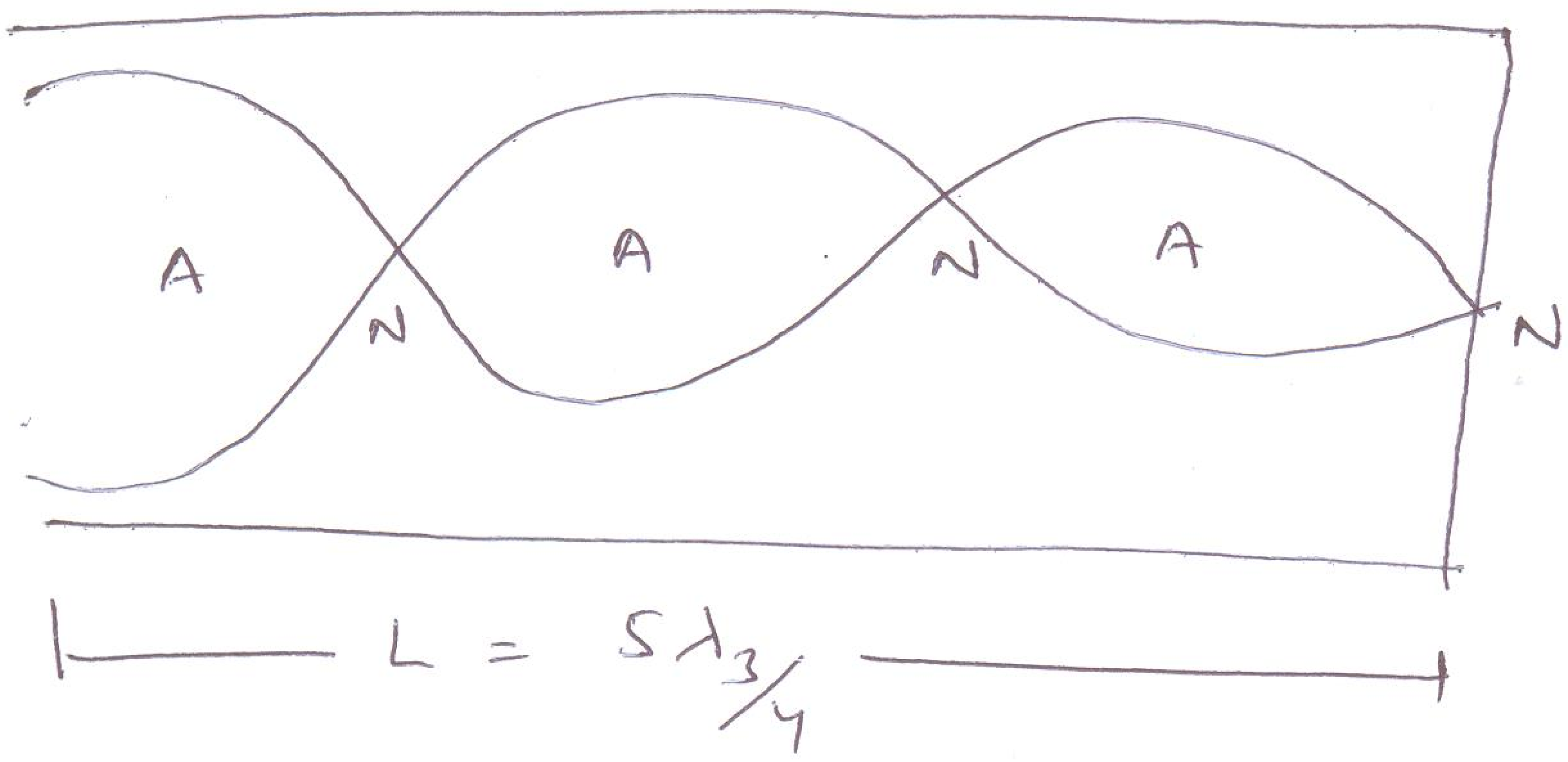
$$\sin \frac{2\pi L}{\lambda} = \pm 1$$

$$\frac{2\pi L}{\lambda} = (2n-1)\frac{\pi}{2}$$

$$\boxed{L = (2n-1)\lambda/4}$$

3rd mode of vibration $n = 3$

$$L = \frac{5\lambda_3}{4} \quad , \quad \lambda_3 = \frac{4L}{5}$$



$$\nu_3 = \frac{5v}{4L} = 5\nu_1$$

⇒ 2nd overtone or 5th harmonic

3.15. FORMATION OF BEATS

The formation of beats can be studied both graphically and analytically as given below :

1. Graphical method. Consider that two trains of harmonic waves of slightly different frequencies (or of slightly different wavelengths) are travelling in a medium. For the sake of simplicity, consider that the two waves are of frequencies $\nu_1 = 50$ Hz and $\nu_2 = 40$ Hz respectively and their amplitudes are equal. In 1/5 second, the two waves will cover the same distance. But in this distance, the number of waves due to the first and the second wave trains will be respectively 10 and 8 as shown in Fig. 3.13 (*a*) and 3.13 (*b*).

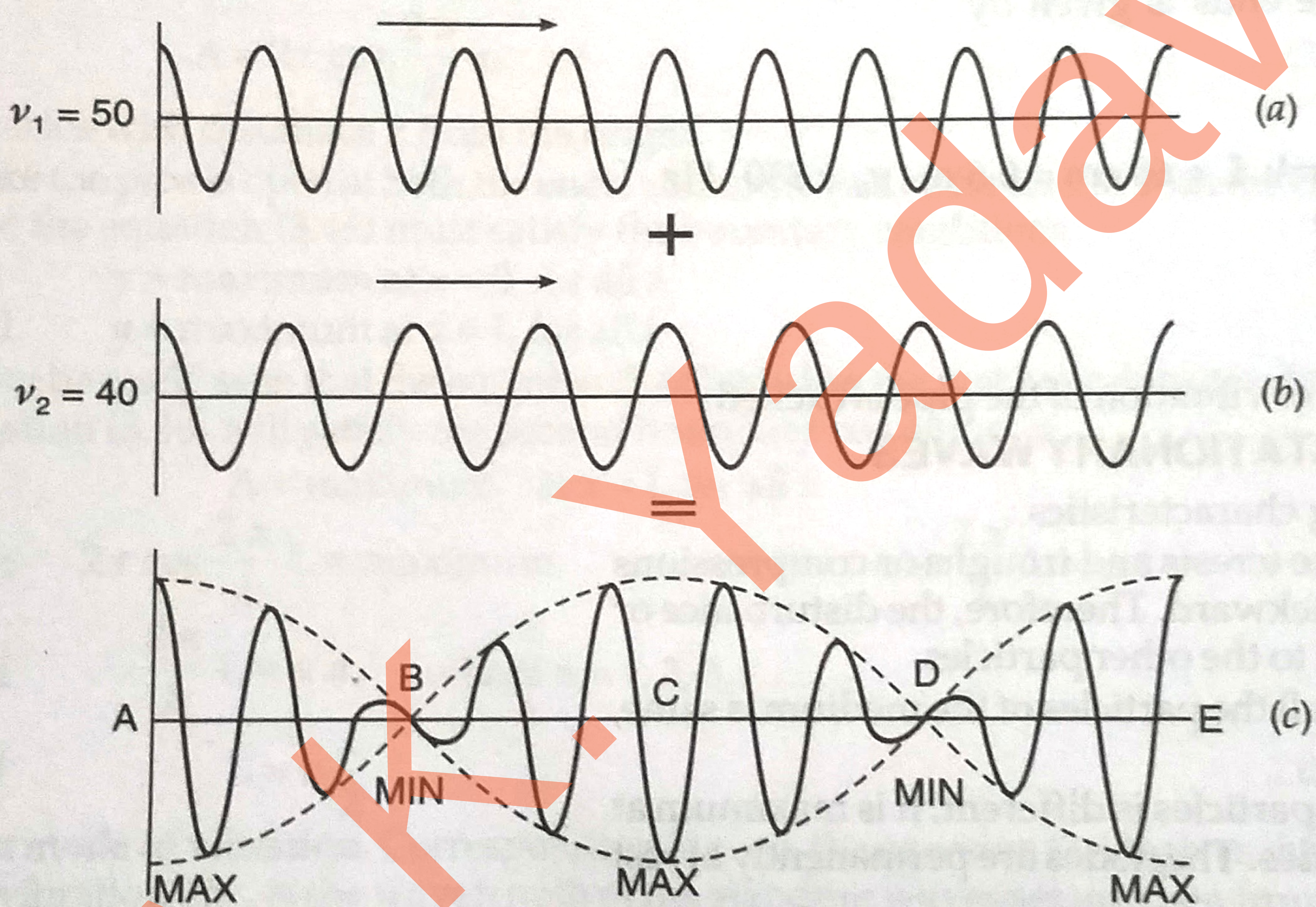


Fig. 3.13

When the two wave trains superpose on each other, the resultant displacement at each point can be found by the principle of superposition *i.e.* by adding the displacements of the two waves at each point according to the law of vector addition. The resultant wave form will be as shown in Fig. 3.13 (*c*). It follows that the amplitude and hence the intensity of the resultant wave becomes maximum, minimum, maximum, minimum and maximum at the points A, B, C, D and E respectively. As such, it follows that two beats are formed, one between A and C and the other between C and E. Since these two beats are formed in 1/5 second, the number of beats formed per second will be 10 *i.e.* equal to the difference in frequencies of the two wave trains.

Analytical method

Consider 2 waves having equal amplitude 'a' & different frequencies ν_1 & ν_2 travelling in a medium in the same direction.

$$y_1 = a \sin \omega_1 t = a \sin 2\pi\nu_1 t$$

$$y_2 = a \sin \omega_2 t = a \sin 2\pi\nu_2 t$$

Acc. to superposition principle

$$y = y_1 + y_2$$

$$= a \sin 2\pi\nu_1 t + a \sin 2\pi\nu_2 t$$

$$= a [\sin 2\pi\nu_1 t + \sin 2\pi\nu_2 t]$$

$$= 2a \cos \pi(\nu_1 - \nu_2)t \cdot \sin \pi(\nu_1 + \nu_2)t$$

$$y = A \sin \pi(\nu_1 + \nu_2)t$$

where $A = 2a \cos \pi(\nu_1 - \nu_2)t$
↳ resultant amplitude

A will be max. when $\cos \pi(\nu_1 - \nu_2)t = \pm 1$

$$\pi(\nu_1 - \nu_2)t = n\pi$$

$$\boxed{t = \frac{n}{\nu_1 - \nu_2}} \quad (1)$$

So, resultant amplitude of wave & hence resultant intensity of sound is max. at times $t = 0, \frac{1}{\nu_1 - \nu_2}, \frac{2}{\nu_1 - \nu_2}, \ldots$

Time interval betn 2 successive maxima $= \frac{1}{\nu_1 - \nu_2} - 0 = \frac{1}{\nu_1 - \nu_2}$

$\therefore$ frequency of maxima $= \nu_1 - \nu_2$ (2)

A will be min. when $\cos \pi(\nu_1 - \nu_2)t = 0$

$$\pi(\nu_1 - \nu_2)t = (2n+1)\frac{\pi}{2}$$

$$\boxed{t = \frac{(2n+1)}{2(\nu_1 - \nu_2)}} \quad (3)$$

So, amplitude of resultant wave & hence resultant intensity of sound is minimum at times $\frac{1}{2(\nu_1-\nu_2)}, \frac{3}{2(\nu_1-\nu_2)}, ----$

Time interval betⁿ 2 successive minima $= \frac{3}{2(\nu_1-\nu_2)} - \frac{1}{2(\nu_1-\nu_2)} = \frac{1}{\nu_1-\nu_2}$

$\therefore$ frequency of minima $= \nu_1 - \nu_2$ — (4)

Comparing of values of t for maxima & minima shows that maximum & minimum intensity of sound occur alternately.

from (2) & (4)

$$\boxed{\text{frequency of beats} = \nu_1 - \nu_2}$$

Applications of beats

(1) It is used in radio reception

(2) It is used by musicians in tuning musical instruments.

(3) It is used in detecting the presence of dangerous gases in mines.

Doppler's effect in sound

Whenever there is a relative motion betⁿ a source of sound & listener, the apparent frequency of sound heard by the listener is different from the actual frequency of sound emitted by the source.

- When the distance between the source & listener is decreasing, the apparent frequency increases & vice-versa.
- Doppler effect is a wave phenomenon.
- It is also applicable to e-m waves.

Expression for apparent frequency

Let S - source of sound initially at rest

L - Listener " " " " "

ν - actual frequency of source emitted by souce

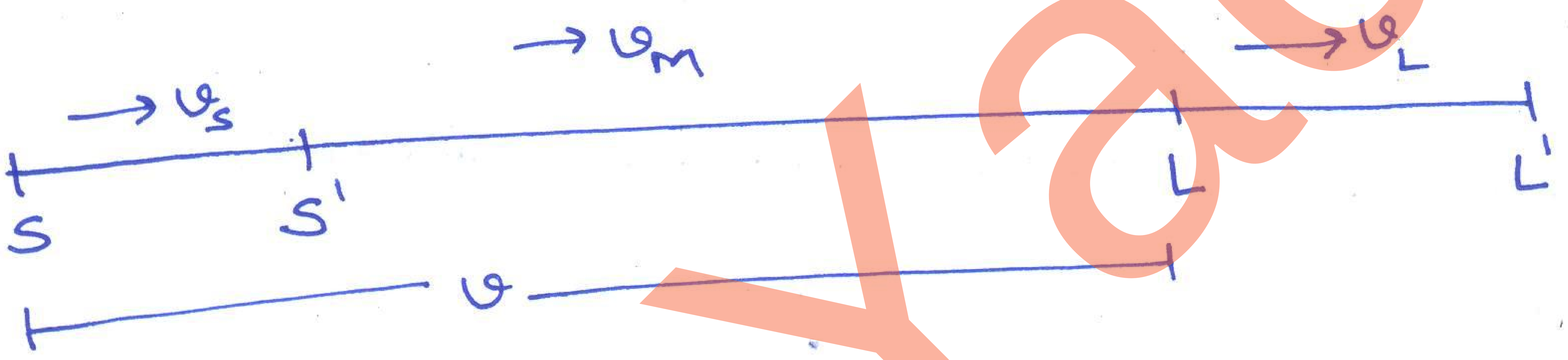
λ - wavelength of emitted sound.

If v is the velocity of sound in still air, then

$$\lambda = \frac{v}{\nu} \quad \text{--- (1)}$$

If the distance bet^n the source & listener is also v then the same frequency i.e. ν is heard by listener also.

Let the medium, source & listener move in the same direction from S to L with velocities v_M, v_s & v_L resp.



Distance travelled by sound in one sec. $= v + v_M$

" " " souce " " " $= v_s$

Now, Distance travelled by sound in one sec. w.r.t souce (Relative velocity of sound w.r.t souce) $= v + v_M - v_s$

$\therefore$ Apparent wavelength of sound wave is

$$\lambda' = \frac{v + v_M - v_s}{\nu} \quad \text{--- (1)}$$

[$\because$ ν remains unchanged by motion of souce or medium]

Distance travelled by listener in one sec. $= v_L$

Relative velocity of sound w.r.t. listener $= v + v_M - v_L$

Apparent frequency of sound wave heard by listened

$$\nu' = \frac{v + v_M - v_L}{\lambda'}$$

$$\boxed{\nu' = \left(\frac{v + v_M - v_L}{v + v_M - v_s}\right)\nu}$$

if the medium is stationary, $v_m = 0$

$$\therefore \quad \boxed{\nu' = \left(\frac{v - v_L}{v - v_s}\right)\nu}$$

Sign convention

1. All velocities along S to L are taken +ve.
2. " " " L to S " " −ve.

Special Cases

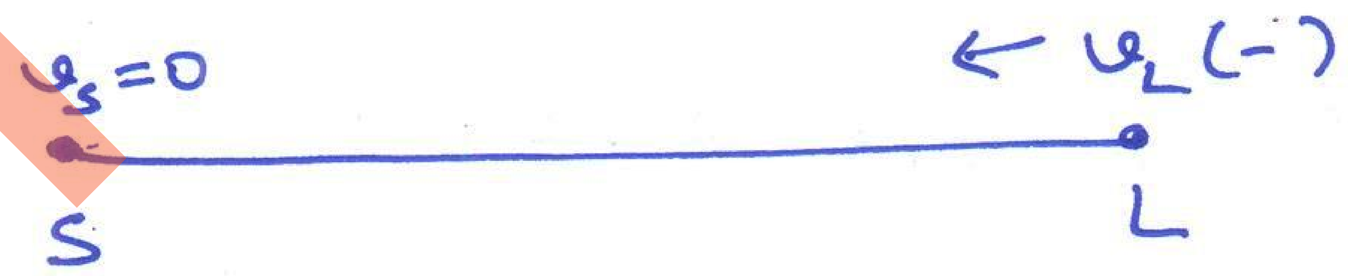
① Source moving towards stationary listener

$$\boxed{\nu' = \left(\frac{v}{v - v_s}\right)\nu} \qquad \nu' > \nu$$

② Source moving away from stationary listener.

$$\boxed{\nu' = \left(\frac{v}{v + v_s}\right)\nu} \qquad \nu' < \nu$$

③ Listener moving towards stationary source



$$\boxed{\nu' = \left(\frac{v + v_L}{v}\right)\nu} \qquad \nu' > \nu$$

④ Listener moving away from stationary source.

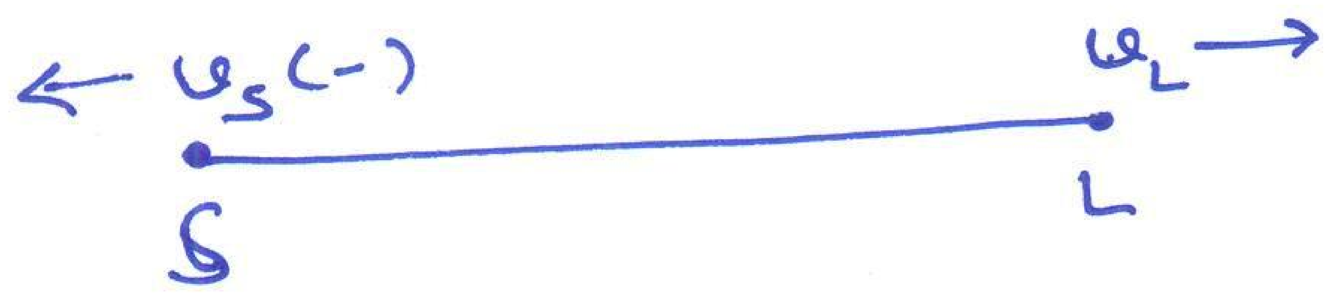
$$\boxed{\nu' = \left(\frac{v - v_L}{v}\right)\nu} \qquad \nu' < \nu$$

⑤ Source & listener moving towards each other

$$\boxed{\nu' = \left(\frac{v + v_L}{v - v_s}\right)\nu}$$

$\nu' > \nu$

⑥ Source & listener moving away from each other



$$\boxed{\nu' = \left(\frac{v - v_L}{v + v_s}\right)\nu}$$

$\nu' < \nu$

⑦ Source & listener moving in same direction with same velocity

$$\boxed{\nu' = \nu}$$

Doppler effect in sound is asymmetric

The apparent frequency of sound when the source moves towards stationary listener is

$$\nu' = \left(\frac{v}{v - v'}\right)\nu \quad \text{——— (1)}$$

The apparent frequency of sound when the listener moves towards stationary source is

$$\nu'' = \left(\frac{v + v'}{v}\right)\nu \quad \text{——— (2)}$$

from (1) & (2)

$$\nu' \neq \nu''$$

i.e. Apparent frequency differs when source or listener approaches the other with same speed.

Hence Doppler effect in sound is asymmetric.