Vectors

Vectors – Those physical quantities which have both magnitude and direction are called vectors. Graphically, a vector can be represented by a straight line with arrow head.

Types of vectors

① **Polar vectors** → Vectors having a starting point or point of application.

eg: Force, displacement

② **Axial vectors** → Vectors which represent rotational effect and act along the axis of rotation.

A few important points

① **Modulus of a vector** – magnitude of a vector

vector → $\vec{A}$, magnitude → $|\vec{A}|$ or A

② **Unit vector** – A unit vector of a given vector is a vector of unit magnitude and has the same direction as that of the given vector.

$$\hat{A} = \frac{\vec{A}}{A}$$

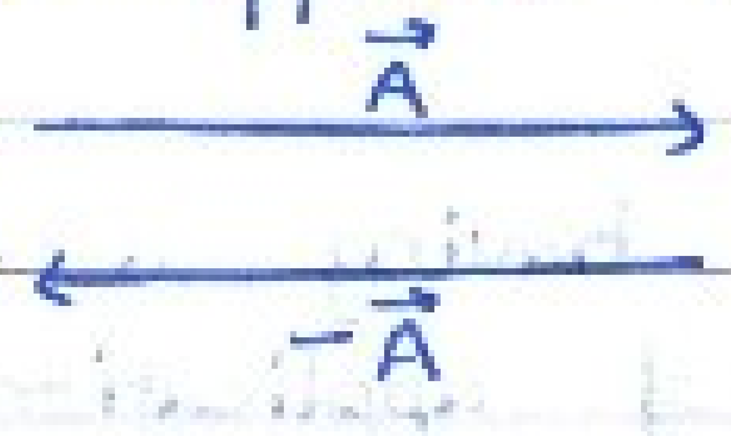
* In cartesian co-ordinates $\hat{i}, \hat{j}$ & $\hat{k}$ are unit vectors along x-axis, y-axis & z-axis respectively.

e.g. $\vec{A} = 4\hat{i} - 3\hat{j} + \hat{k}$, $A = \sqrt{4^2+(-3)^2+(1)^2} = \sqrt{26}$

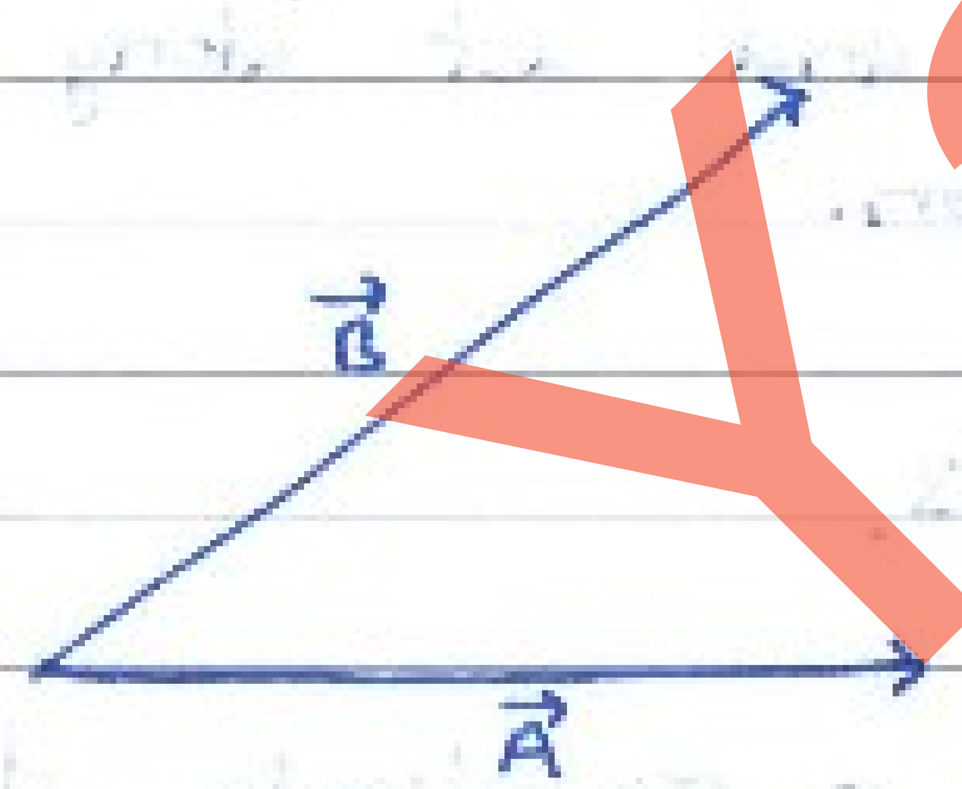
$$\hat{A} = \frac{\vec{A}}{A} = \frac{4\hat{i} - 3\hat{j} + \hat{k}}{\sqrt{26}} = \frac{4}{\sqrt{26}}\hat{i} - \frac{3}{\sqrt{26}}\hat{j} + \frac{1}{\sqrt{26}}\hat{k}$$

③ **Equal vectors** – Two vectors are said to be equal if they have equal magnitude & same direction.

④ **Negative vector** – A negative vector of a given vector is a vector of same magnitude but acting in a direction opposite to that of given vector.

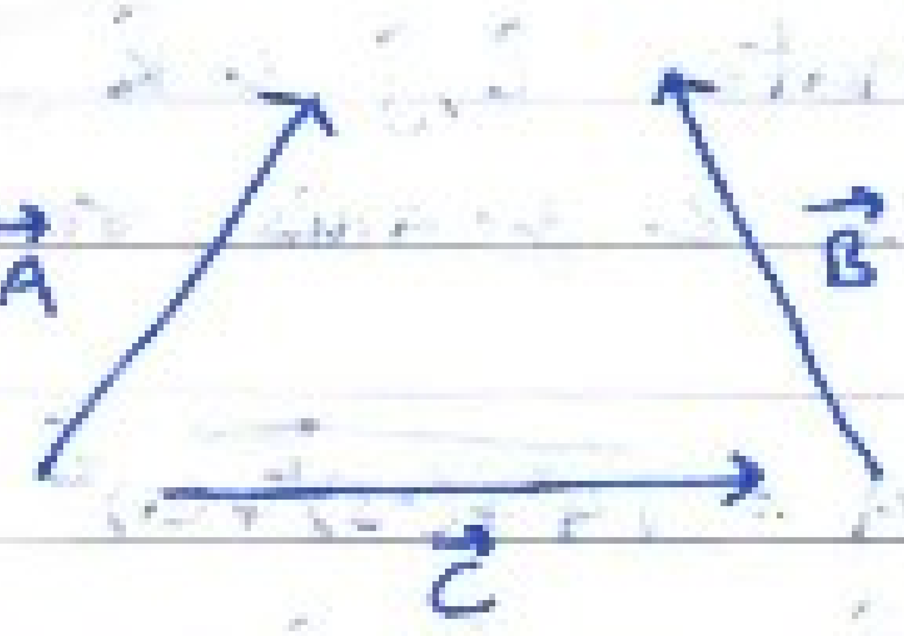


⑤ **Co-initial vectors** – Vectors are said to be co-initial if their initial point is common.

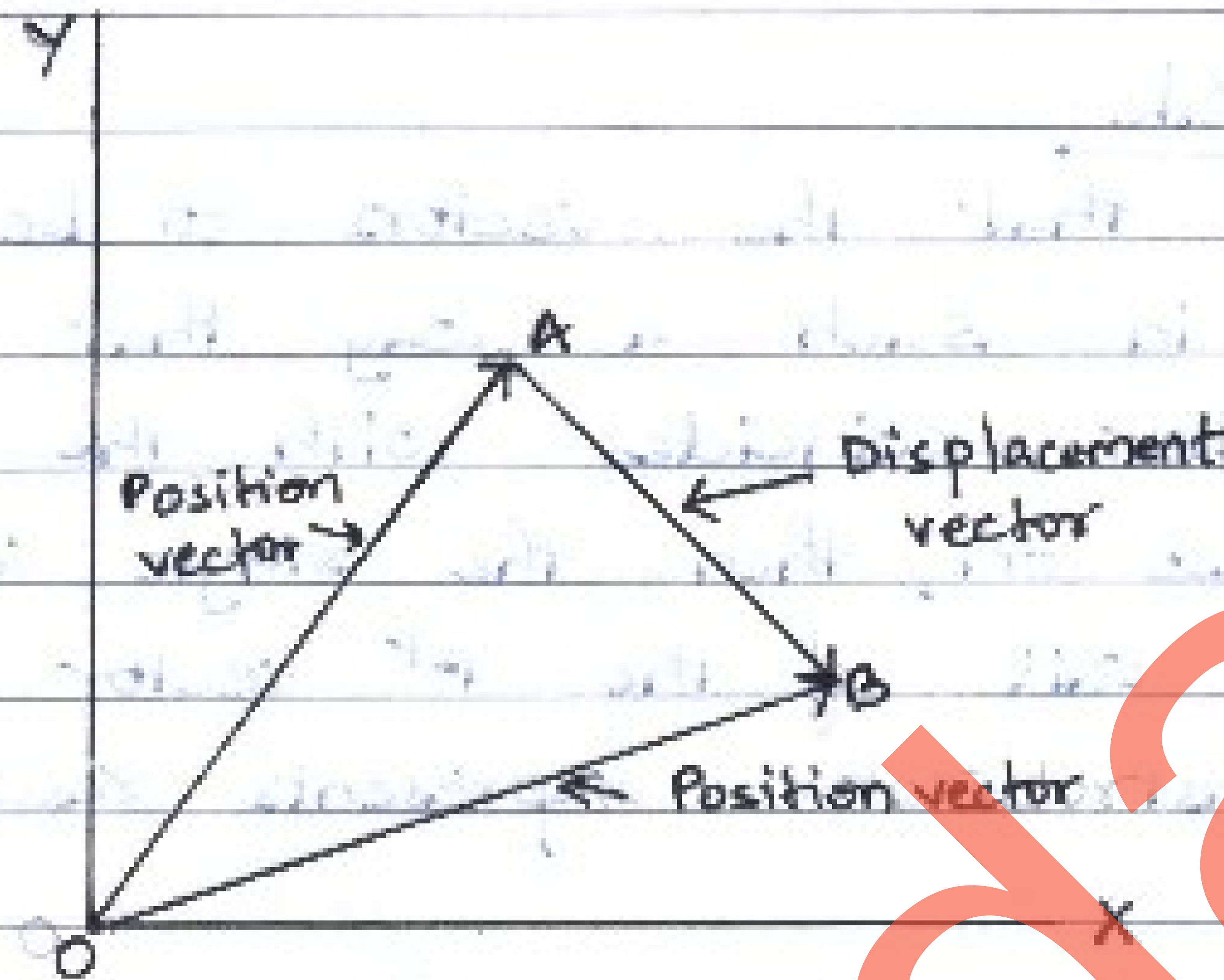


⑥ **Collinear vectors** – Vectors which are having equal or unequal magnitudes & are acting along parallel straight lines.

⑦ **Coplanar vectors** – Vectors acting in the same plane.



Position vector & displacement vector



Multiplication of a vector

(a) By a real number

$n(\vec{A}) = n\vec{A}$ $\qquad$ $\xrightarrow{\vec{A}}$ $\qquad$ $\xrightarrow{2\vec{A}}$

$-n(\vec{A}) = -n\vec{A}$ $\qquad$ $\xrightarrow{\vec{A}}$ $\qquad$ $\xleftarrow{-2\vec{A}}$

(b) By a scalar

S (scalar) multiplied by $\vec{A}$ = $S\vec{A}$ (vector)

eg $\vec{A}$ = 100 N due west

S = 10 sec.

$S\vec{A}$ = 10 x 100

= 1000 N-s due west.

Vector addition by Geometrical method

General rule

It states that the vectors to be added are arranged in such a way that the head of first vector coincides with the tail of 2^{nd} vector & so on, then the single vector drawn from the tail of the 1^{st} vector to the head of the last vector represents resultant vector.

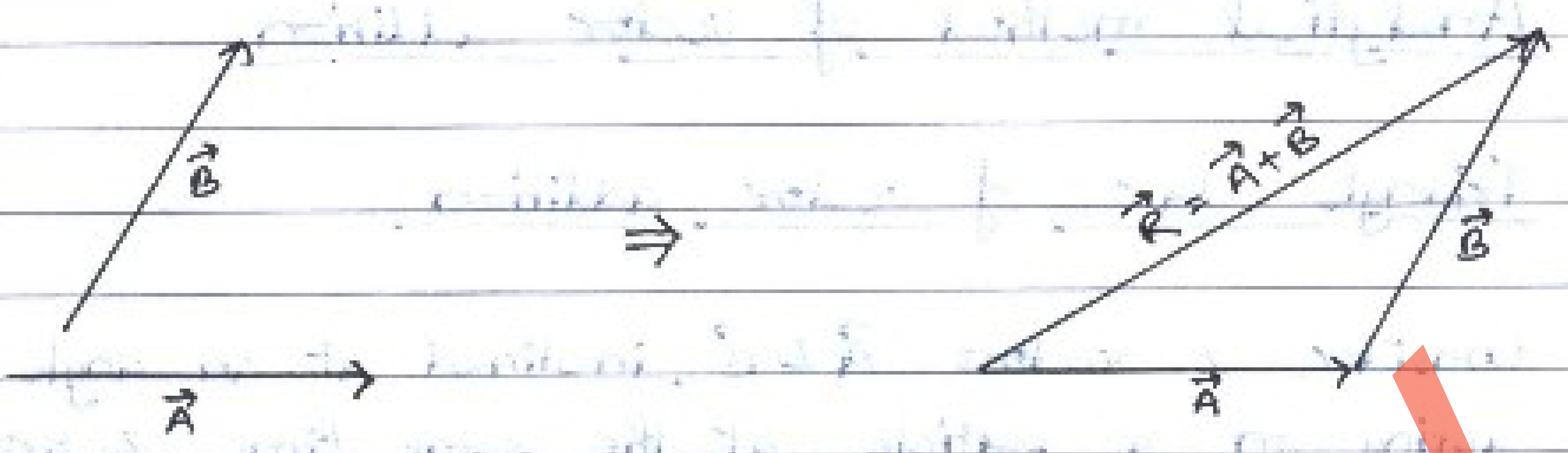
Case 1 → 2 vectors acting in same direction

Case 2 → 2 vectors acting in opposite direction

Case 3 → 2 vectors act at some angle

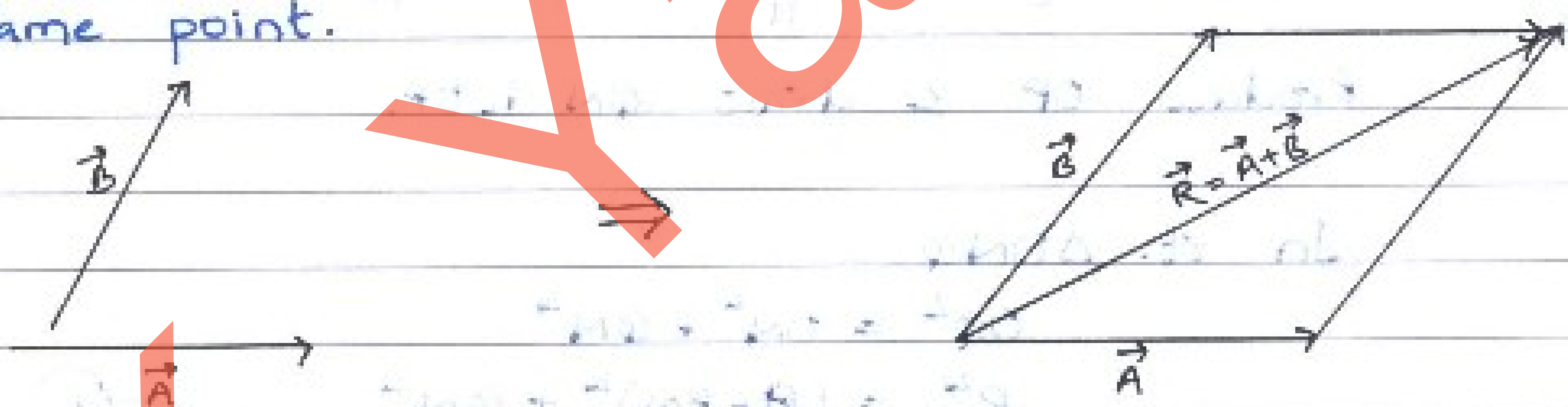
(a) **Triangle law of vectors**

If 2 vectors acting on a particle at the same time are represented in magnitude and direction by the 2 sides of a triangle taken in one order, their resultant vector is represented in magnitude and direction by the 3^{rd} side of the triangle taken in opposite order.

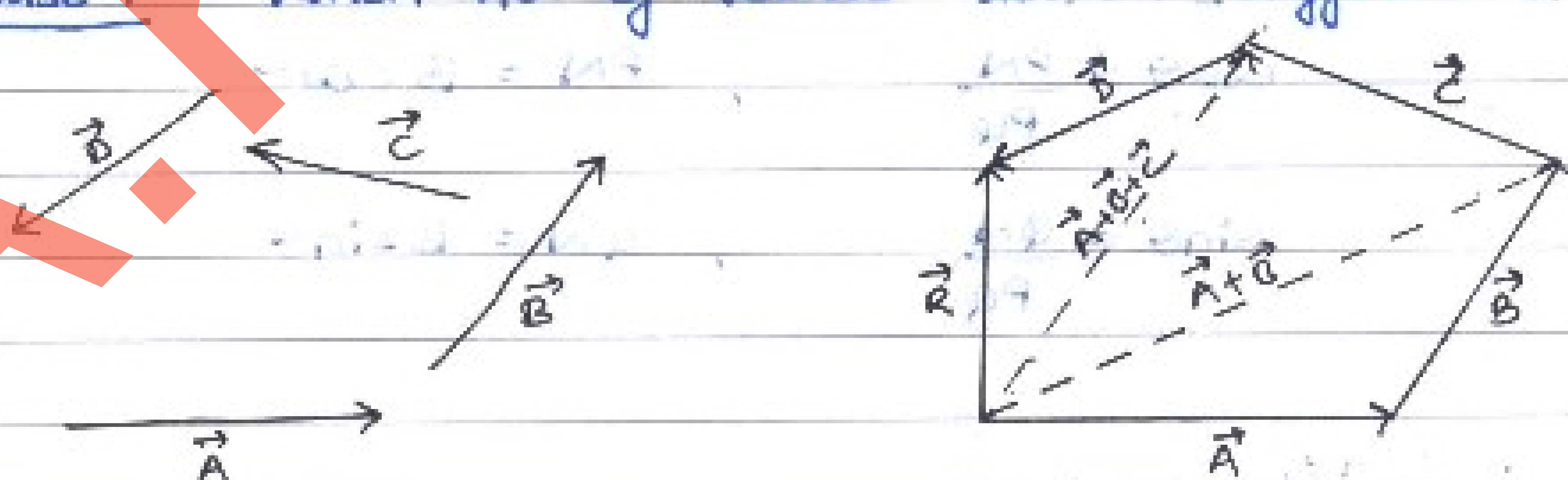


(b) Parallelogram law of vectors

If 2 vectors acting on a particle at the same time are represented in magnitude & direction by the 2 adjacent sides of a parallelogram drawn from a point, their resultant vector is represented in magnitude & direction by the diagonal of the parallelogram drawn from the same point.



Case 4 → When no. of vectors act in different directions.



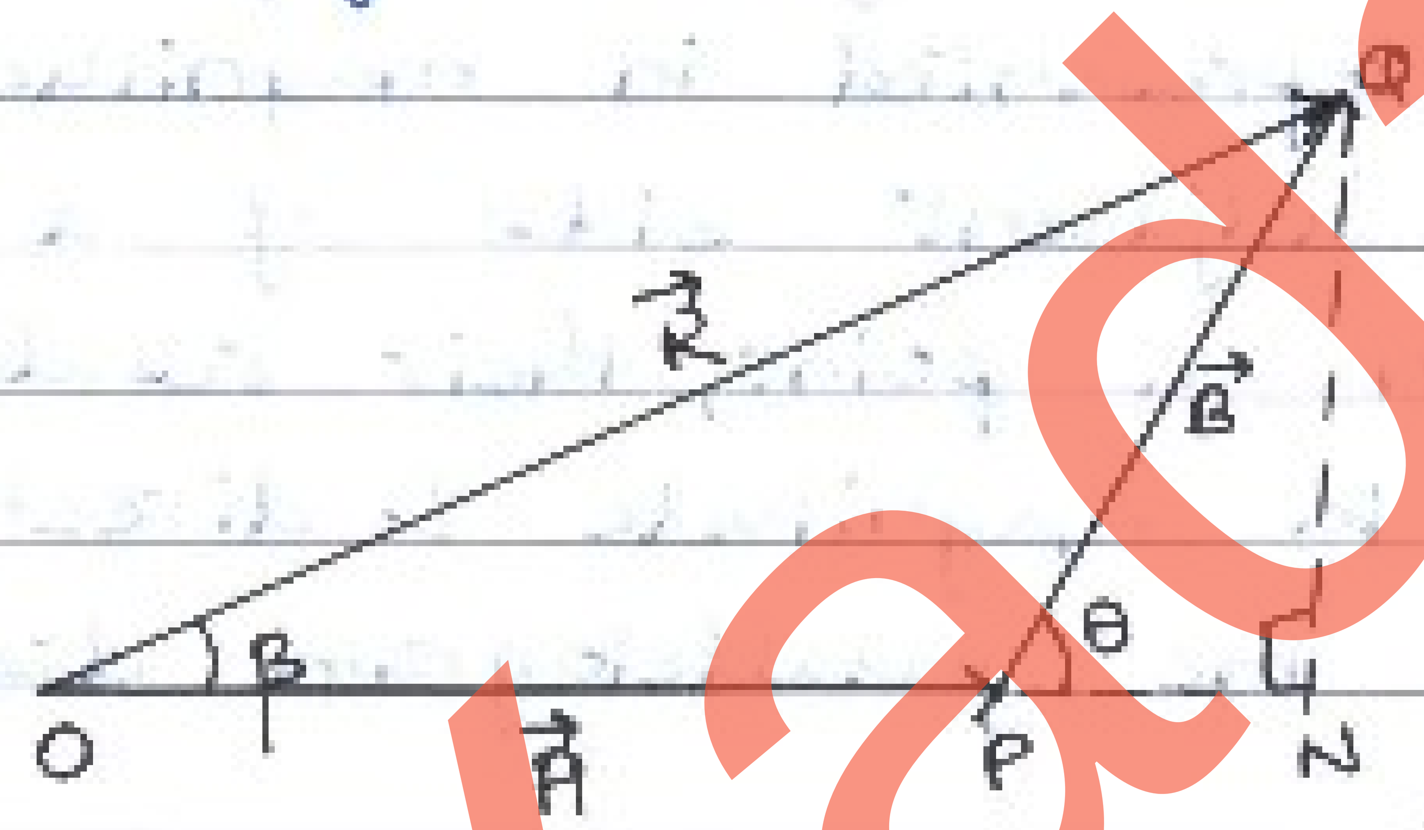
Polygon law of vectors

If any no. of vectors, acting on a particle at the same time are represented in magnitude & direction by various sides of an open polygon taken in the same order, their resultant is represented in magnitude and direction by the closing side of polygon taken in opp order.

Analytical method of vector addition

① Triangle law of vector addition

Consider 2 vectors $\vec{A}$ & $\vec{B}$, inclined at an angle θ be acting on a particle at the same time & are represented in magnitude & direction by the sides $\vec{OP}$ & ~~OQ~~ $\vec{PQ}$ of ΔOPQ taken in same order.



Produce OP & draw $QN \perp OP$.

In rt. ΔONQ

$$OQ^2 = ON^2 + QN^2$$

$$R^2 = (A + PN)^2 + QN^2 \quad \text{—①}$$

In rt. ΔPNQ

$$\cos\theta = \frac{PN}{PQ}, \quad PN = B\cos\theta$$

$$\sin\theta = \frac{QN}{PQ}, \quad QN = B\sin\theta$$

$\therefore$ eqⁿ ① becomes

$$R^2 = (A + B\cos\theta)^2 + (B\sin\theta)^2$$

$$= A^2 + B^2\cos^2\theta + 2AB\cos\theta + B^2\sin^2\theta$$

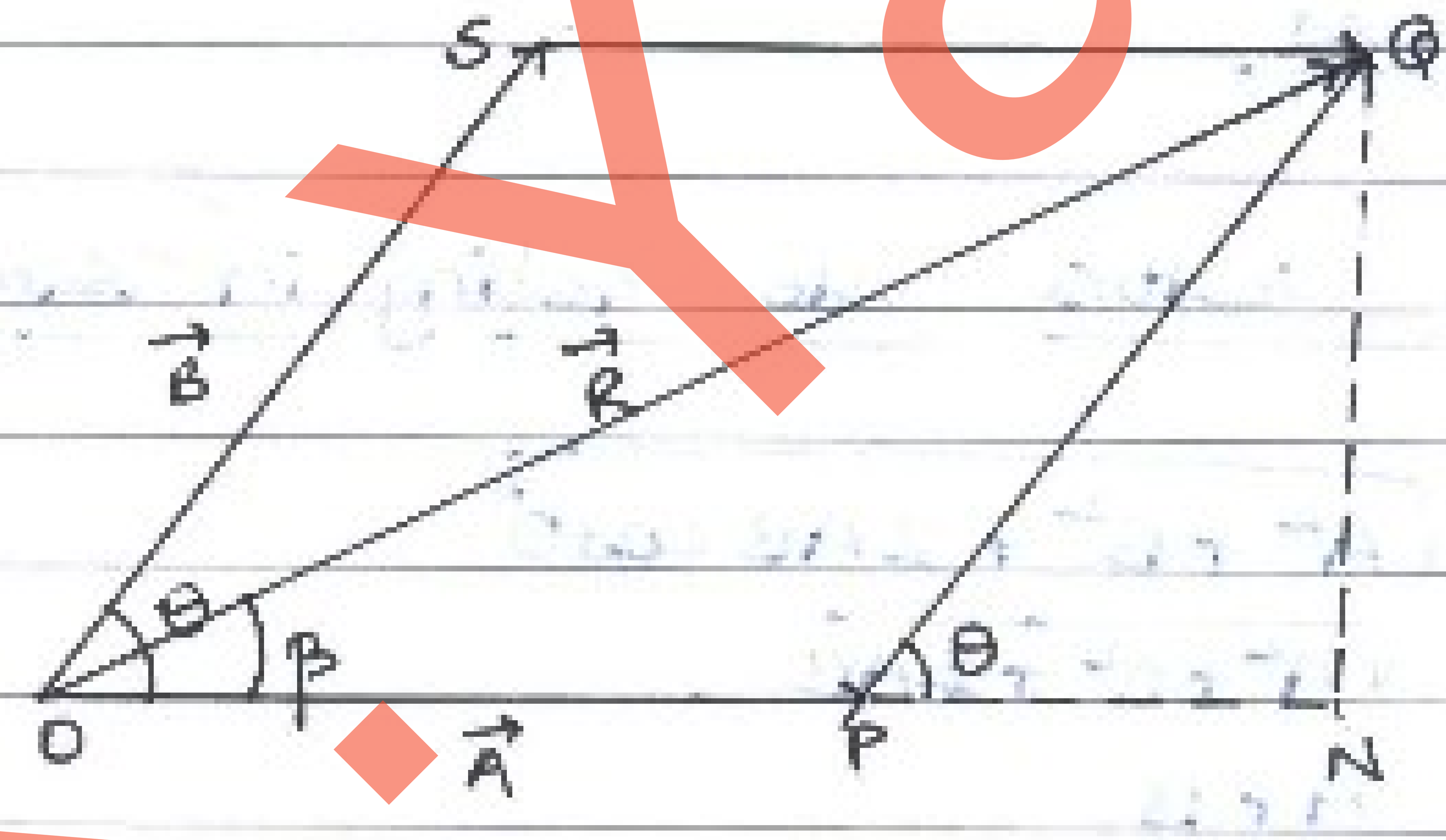
$$\boxed{R = \sqrt{A^2 + B^2 + 2AB\cos\theta}}$$

Let the resultant $\vec{R}$ makes an angle β with direction of $\vec{A}$.

$$\boxed{\tan\beta = \frac{QN}{ON} = \frac{B\sin\theta}{A + B\cos\theta}}$$

② Parallelogram law of vector addition

Consider 2 vectors $\vec{A}$ & $\vec{B}$ inclined at an angle θ be acting on a particle at the same time and are represented in magnitude & direction by two adjacent sides $\vec{OP}$ & $\vec{OS}$ of parallelogram OPQS drawn from point O.



Produce ~~ON~~ OP & draw $QN \perp OP$

In rt. ΔONQ

$$OQ^2 = ON^2 + QN^2$$

$$R^2 = (A + PN)^2 + QN^2$$

In rt. ΔPNQ

$$\cos\theta = \frac{PN}{PQ} , \quad PN = B\cos\theta$$

$$\sin\theta = \frac{QN}{PQ} , \quad QN = B\sin\theta$$

$\therefore$ eq^n ① becomes

$$R^2 = (A + B\cos\theta)^2 + B^2\sin^2\theta$$
$$= A^2 + B^2\cos^2\theta + 2AB\cos\theta + B^2\sin^2\theta$$

$$\boxed{R = \sqrt{A^2 + B^2 + 2AB\cos\theta}}$$

Let the resultant $\vec{R}$ makes an angle β with the direction of $\vec{A}$.

$$\boxed{\tan\beta = \frac{QN}{ON} = \frac{B\sin\theta}{A + B\cos\theta}}$$

Special cases

(i) When the vectors are acting in same direction ($\theta = 0°$)

$$R = \sqrt{A^2 + B^2 + 2AB\cos 0°}$$
$$= \sqrt{A^2 + B^2 + 2AB}$$
$$R = A + B$$

$$\tan\beta = \frac{B\sin 0°}{A + B\cos 0°} = 0°$$

(ii) opp. direction ($\theta = 180°$)

$$R = \sqrt{A^2 + B^2 - 2AB} = A - B$$
$$\tan\beta = \frac{B \times 0}{A + B(-1)} = 0° \text{ or } 180°$$

(iii) Vectors $\perp^r$ to each other

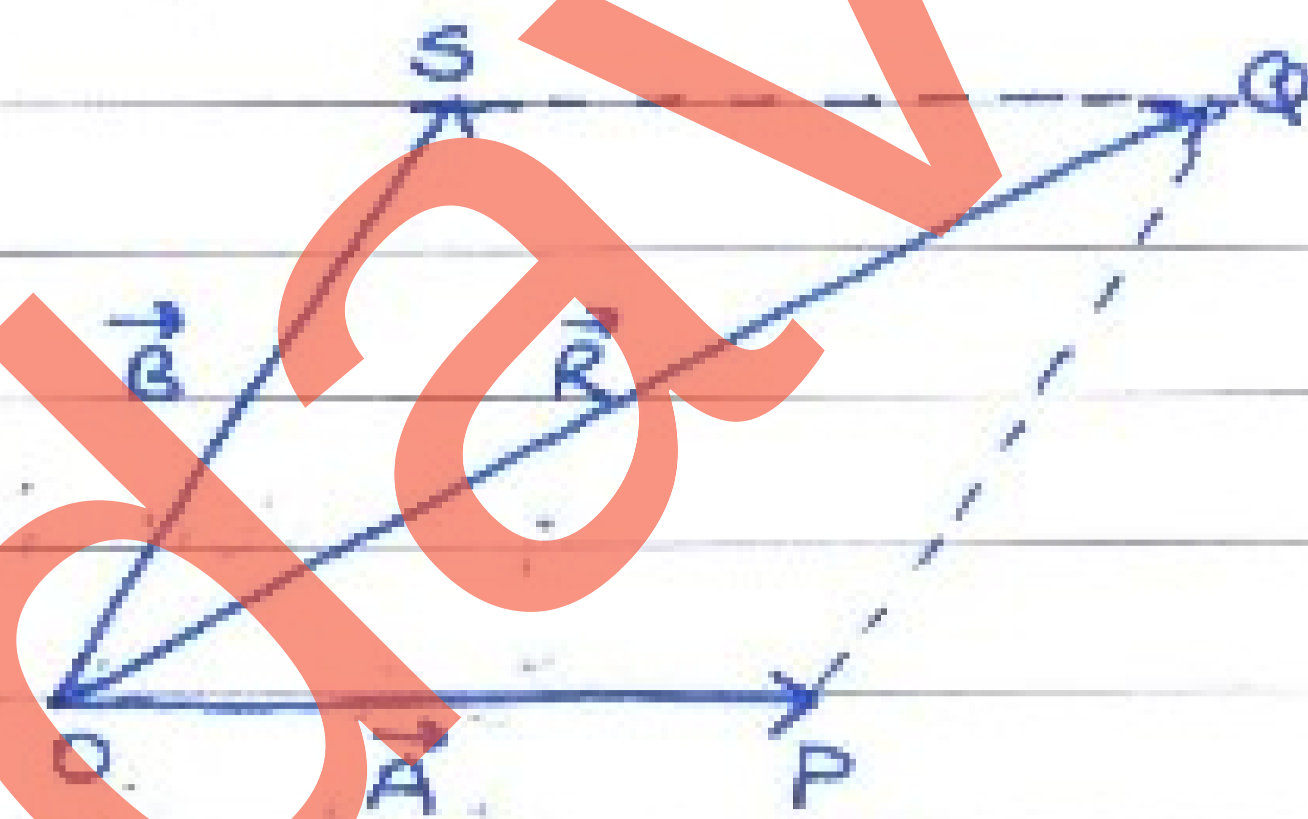
$R = \sqrt{A^2 + B^2}$, $\tan\beta = \frac{B}{A} \Rightarrow \beta = \tan^{-1}\left(\frac{B}{A}\right)$

Points to remember

1. Vectors of the same nature alone can be added.
2. Vector addition is commutative.

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

Proof:



In ΔOPQ

$$\vec{R} = \vec{A} + \vec{B} \quad \text{--- (1)}$$

In ΔOSQ

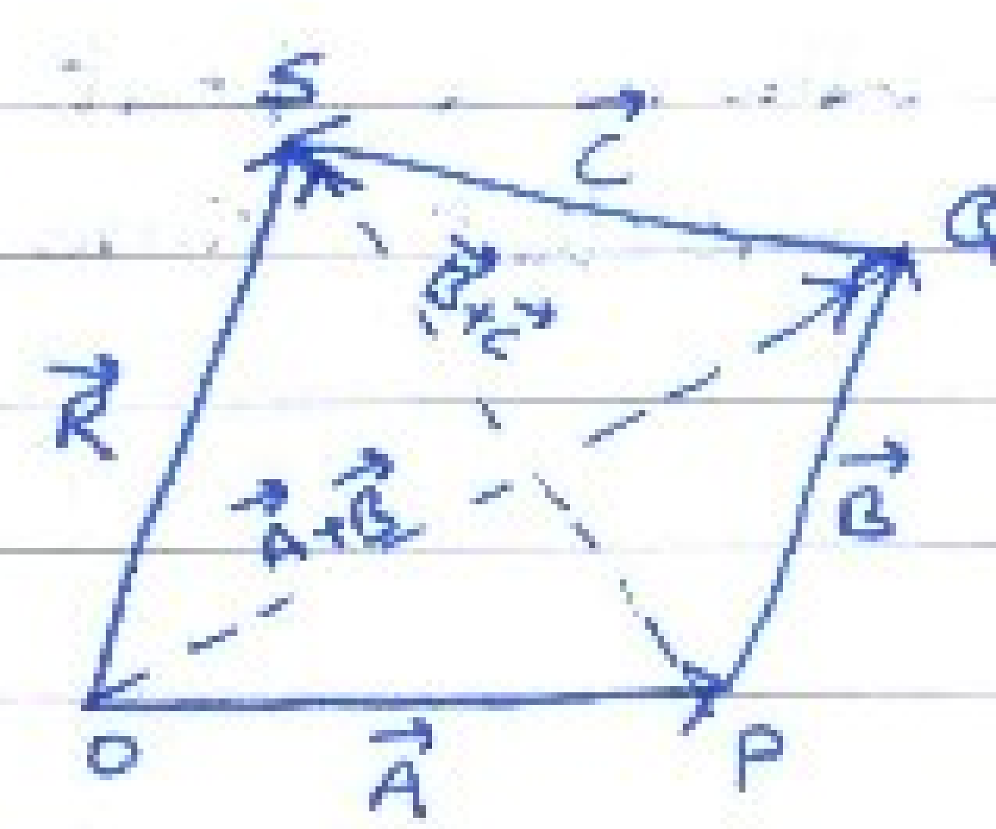
$$\vec{R} = \vec{B} + \vec{A} \quad \text{--- (2)}$$

from (1) & (2) $\boxed{\vec{A} + \vec{B} = \vec{B} + \vec{A}}$

3. Vector addition is associative

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$$

Proof:



In ΔOQS

$$\vec{OS} = \vec{R} = \vec{OQ} + \vec{QS}$$

$$\vec{R} = (\vec{A} + \vec{B}) + \vec{C} \quad \text{--- (1)}$$

In ΔOPS

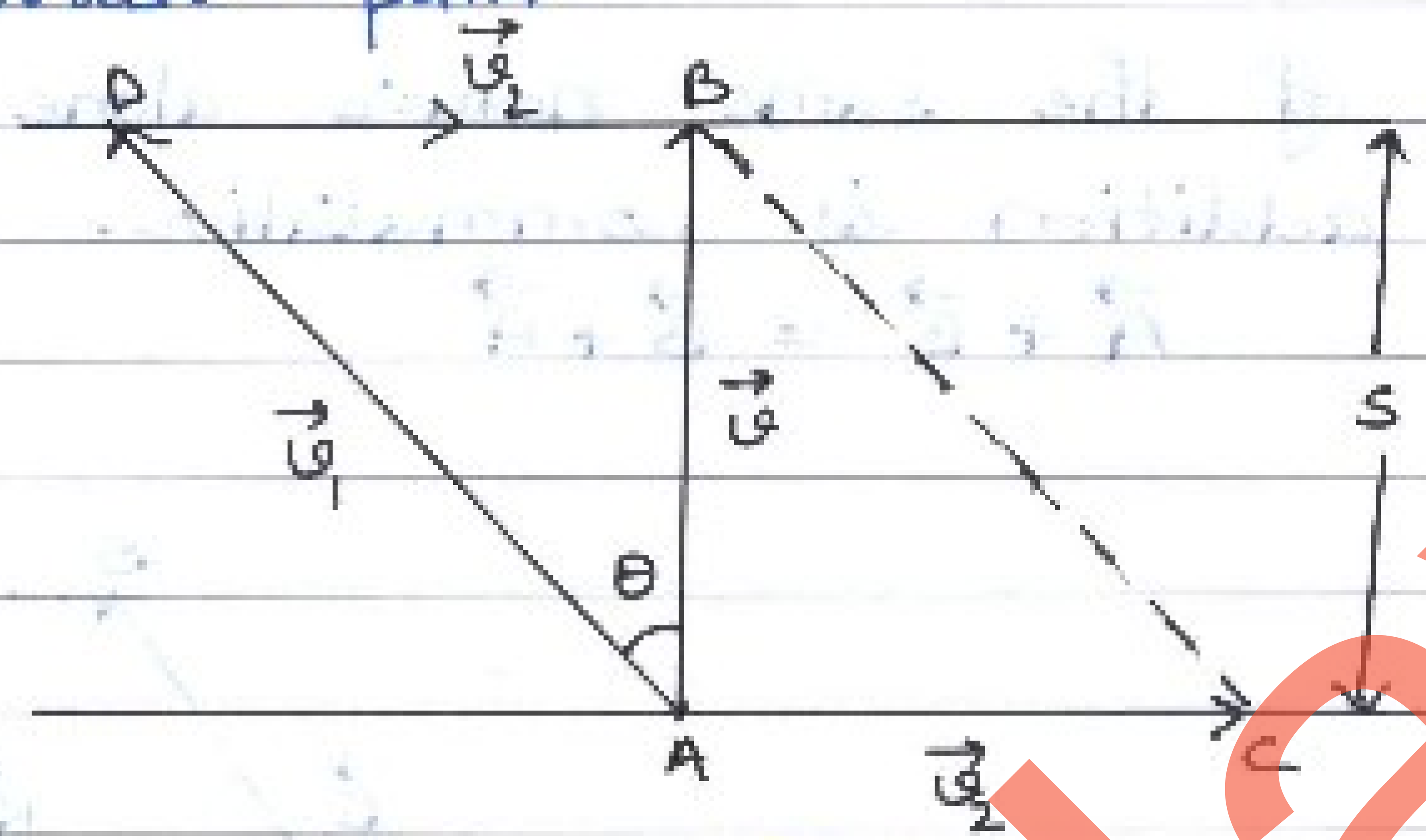
$$\vec{OS} = \vec{OP} + \vec{PS}$$

$$\vec{R} = \vec{A} + (\vec{B} + \vec{C}) \quad \text{--- (2)}$$

from (1) & (2)

$$\boxed{(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})}$$

(4) When a boat tends to cross a river along a shortest path



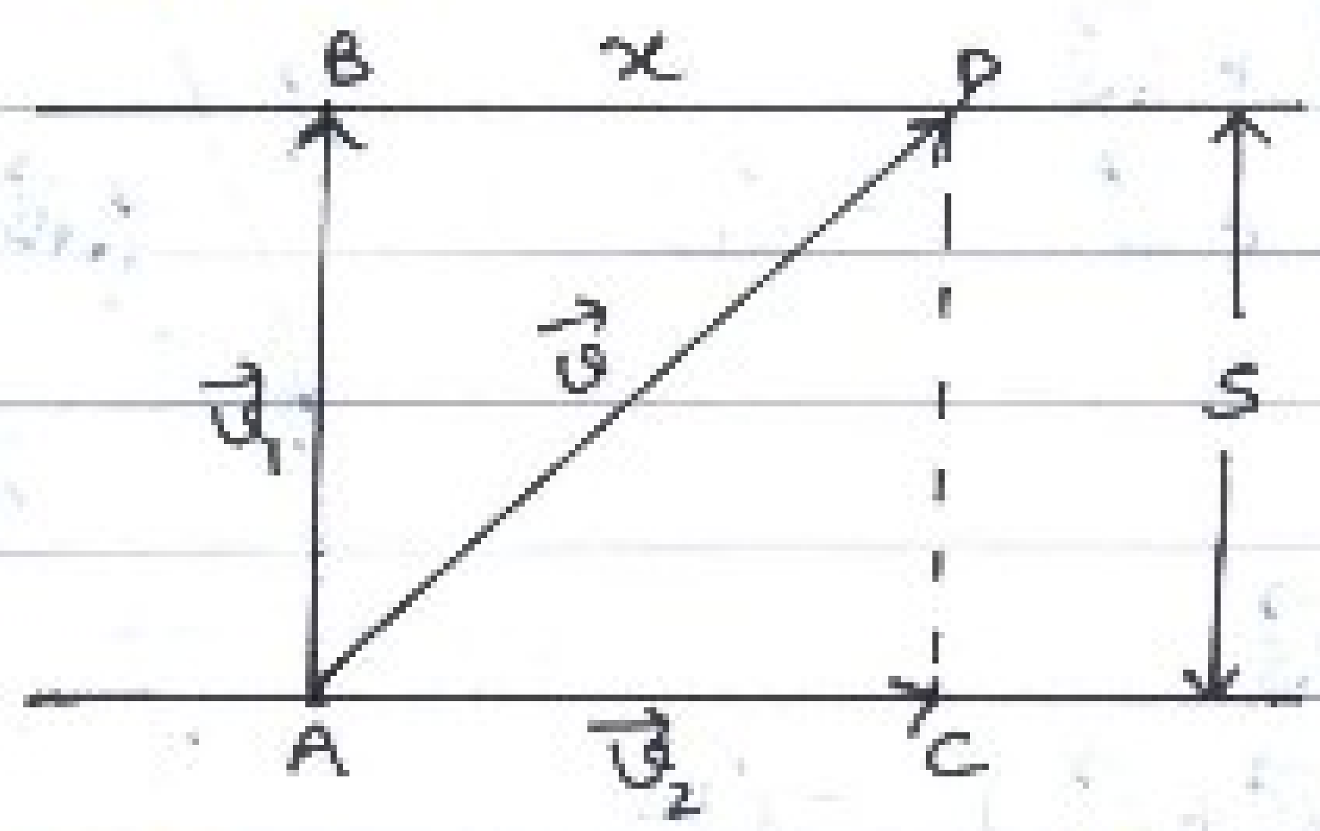
v_1 – velocity of boat
v_2 – " " river
v → resultant velocity

$$v = \sqrt{v_1^2 - v_2^2}$$

$$\sin\theta = \frac{v_2}{v_1}$$

time of crossing the river, $t = \frac{s}{v} = \frac{s}{\sqrt{v_1^2 - v_2^2}}$

(5) When a boat tends to cross a river in shortest time



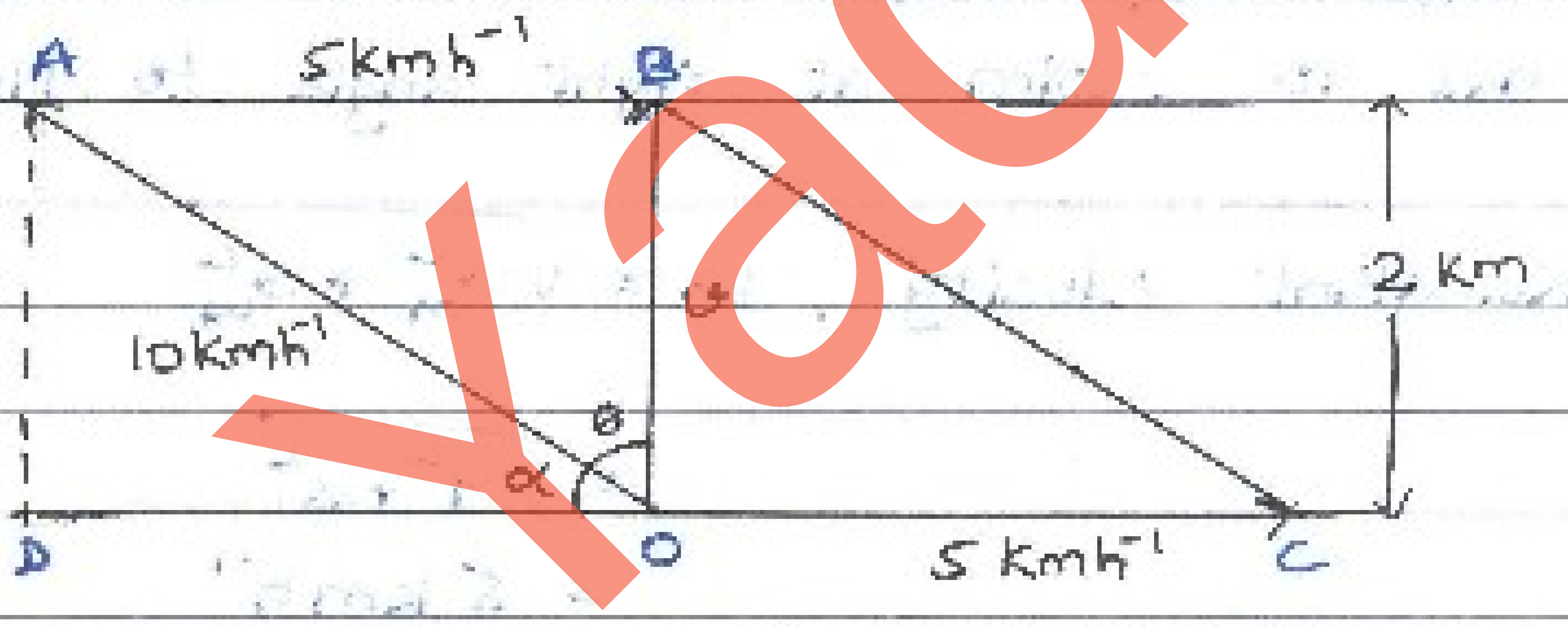
$v = \sqrt{v_1^2 + v_2^2}$, $\tan\theta = \frac{v_2}{v_1}$

time of crossing, $t = \frac{AD}{v} = \frac{\sqrt{s^2 + x^2}}{\sqrt{v_1^2 + v_2^2}}$

Numerical on crossing the river

1.	A boatman can row with a speed of $10 kmh^{-1}$ in still water. If the river flows steadily at $5 kmh^{-1}$, in which direction should the boatman row in order to reach a point on the other bank directly opposite to the point from where he started? The width of the river is 2 km.
2.	A river is flowing from west to east at a speed of $5 ms^{-1}$. A man on the south bank of the river, capable of swimming at $10 ms^{-1}$ in still water, wants to swim across the river in shortest time. In what direction, should he swim?
3.	The stream of the river Jamuna is flowing with a speed of $1 kmh^{-1}$. What should be the direction of the swimmer to cross the river straight, his speed being $2 kmh^{-1}$?
4.	A river one kilometer wide is flowing at $3 kmh^{-1}$, A swimmer, whose velocity in still water is $4 kmh^{-1}$, can swim only for 15 minutes. In what direction should he strike out so as to reach the opposite bank in 15 minutes? What total distance will he swim?
5.	A swimmer crosses a flowing river of width 'd' to and fro in time t1.. The time taken to cover the same distance up and down the stream is t2.If t3 is the time the swimmer would take to swim a distance 2d in still water, then prove that $(t_1)^2 = t_2 t_3$

Ans-1



Method I

In ΔODA

$$\cos\alpha = \frac{OD}{AO} = \frac{5}{10} = \frac{1}{2}$$

$$\alpha = 60^\circ$$

$\therefore$ Boatman should row at an angle $(180 - 60 = 120^\circ)$ with the direction of flow of river.

Method II

In ΔOBA

$$\sin\theta = \frac{AB}{AO} = \frac{5}{10} = \frac{1}{2}$$

$$\theta = 30^\circ$$

$\therefore$ Boatman should row at an angle $(90 + 30 = 120^\circ)$ with the direction of flow of the river

Ans-2 Same as Ans-1

Ans-3 Same as Ans-1

Ans-4 $v_m = 4\, kmh^{-1}$

Distance covered by swimming for 15 min. $(\frac{15}{60} hr)$

$$S = \frac{1}{4} \times 4 = 1\, km$$

Since the river's width is also 1 km, so he has to swim at right angle to flow of river.

Resultant velocity, $v = \sqrt{v_m^2 + v_w^2}$

$$= \sqrt{4^2 + 3^2}$$

$$= 5\, kmh^{-1}$$

Total distance $= 5 \times \frac{1}{4} = 1.25\, km.$

Ans-5 Let u - velocity of swimmer in still water

v - " " " in river "

$$t_1 = \frac{2d}{\sqrt{u^2 - v^2}}$$

$$t_2 = \frac{d}{u+v} + \frac{d}{u-v}$$

$$= \frac{2ud}{u^2 - v^2}$$

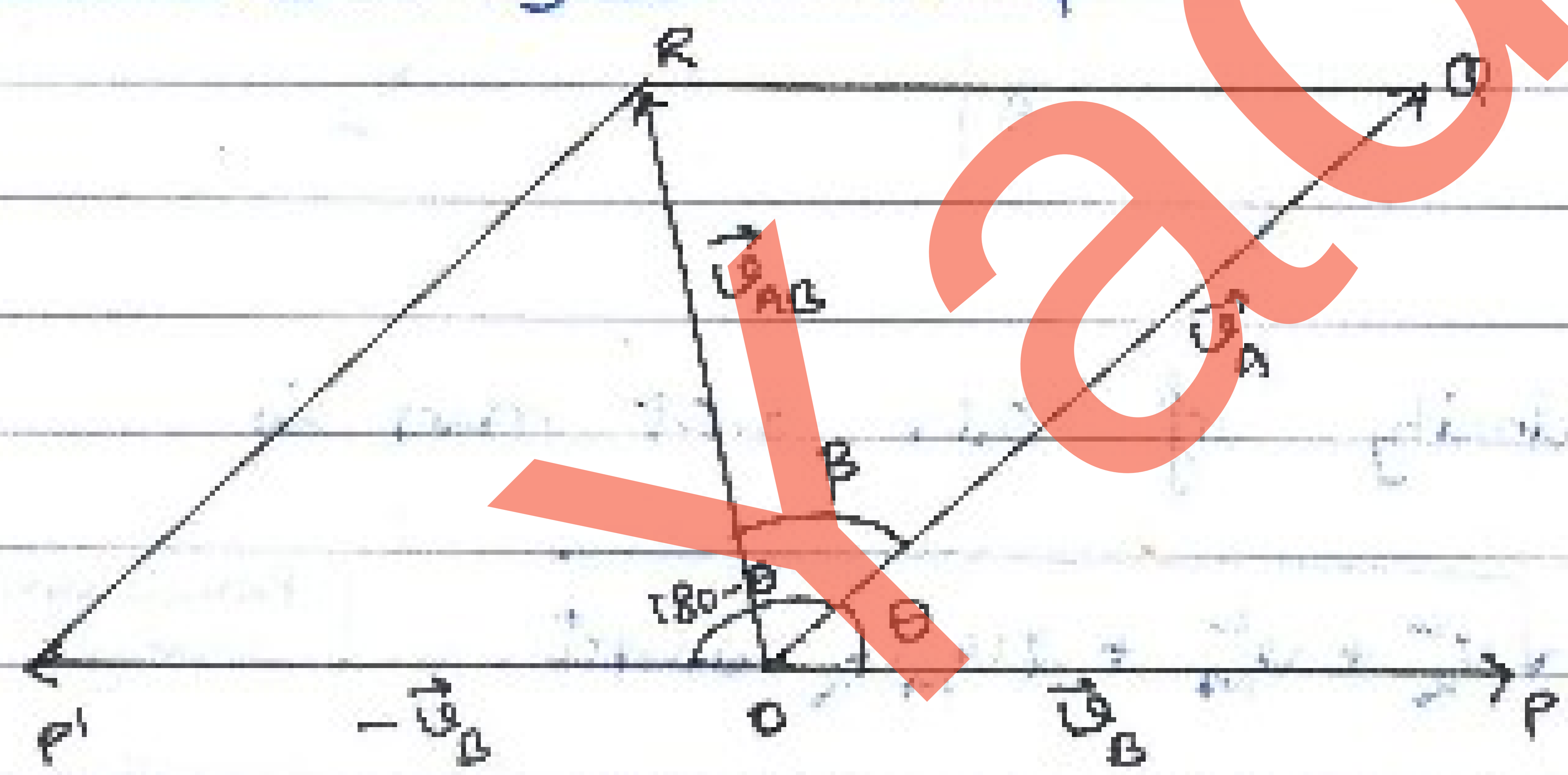
$t_3 = \frac{2d}{u}$

$$t_2 t_3 = \frac{2ud}{u^2 - v^2} \times \frac{2d}{u}$$

$$= \frac{4d^2}{u^2 - v^2}$$

$$= t_1^2$$

Relative velocity in a plane



In parallelogram OQRP'

$$v_{AB} = \sqrt{v_A^2 + v_B^2 + 2 v_A v_B \cos(180 - \theta)}$$ [Using parallelogram law of vector addition]

$$= \sqrt{v_A^2 + v_B^2 - 2 v_A v_B \cos\theta}$$

If $\vec{v}_{AB}$ makes an angle β with the direction of $\vec{v}_A$ then

$$\tan\beta = \frac{v_B \sin(180 - \theta)}{v_A + v_B \cos(180 - \theta)}$$

$$= \frac{v_B \sin\theta}{v_A - v_B \cos\theta}$$

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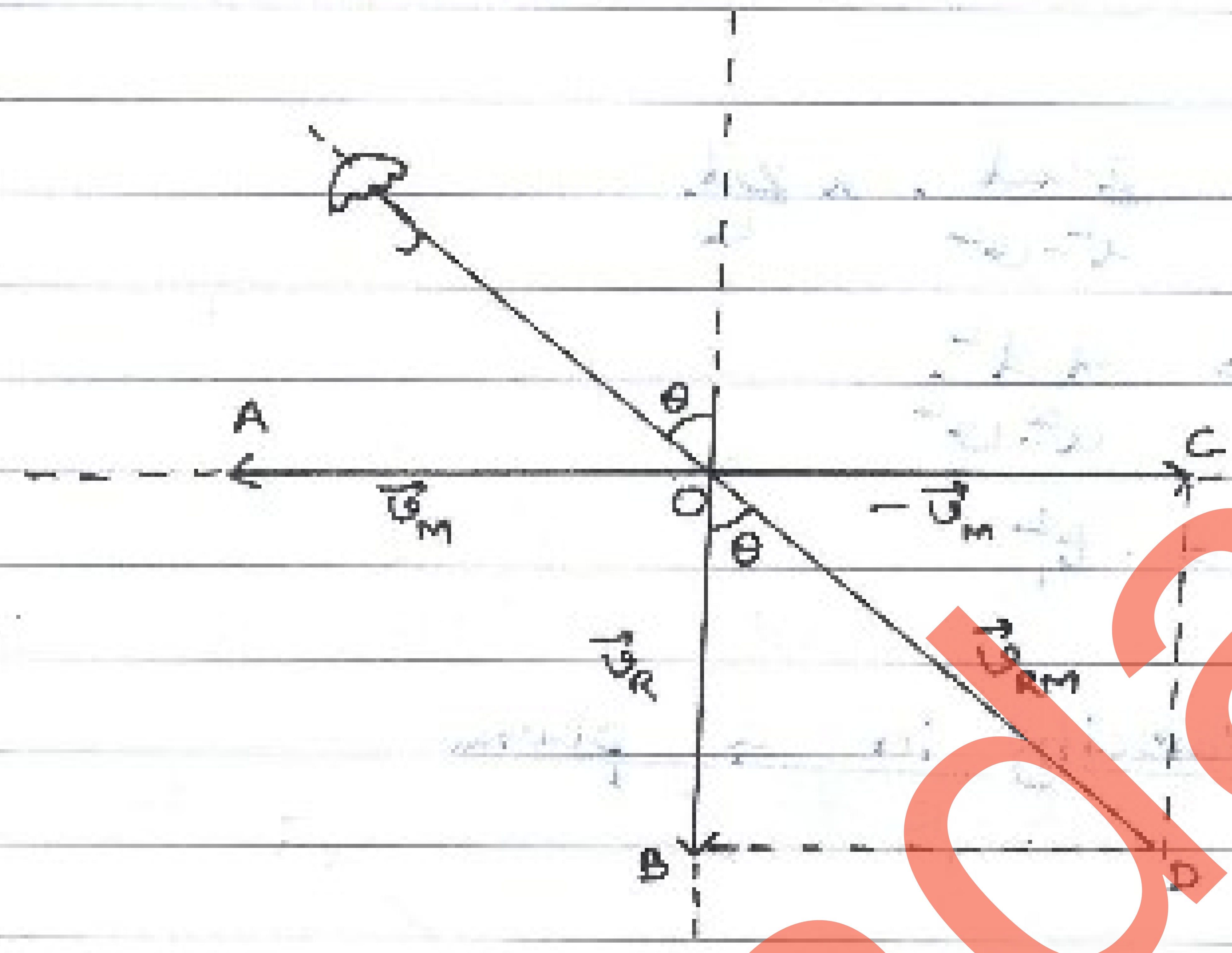
Relative velocity of rain w.r.t moving man



Relative velocity of rain w.r.t man is

$$\vec{v}_{RM} = \sqrt{v_R^2 + v_M^2 + 2v_M v_R \cos 90^\circ}$$ [Parallelogram law in rectangle OBDC]

$$= \sqrt{v_R^2 + v_M^2}$$

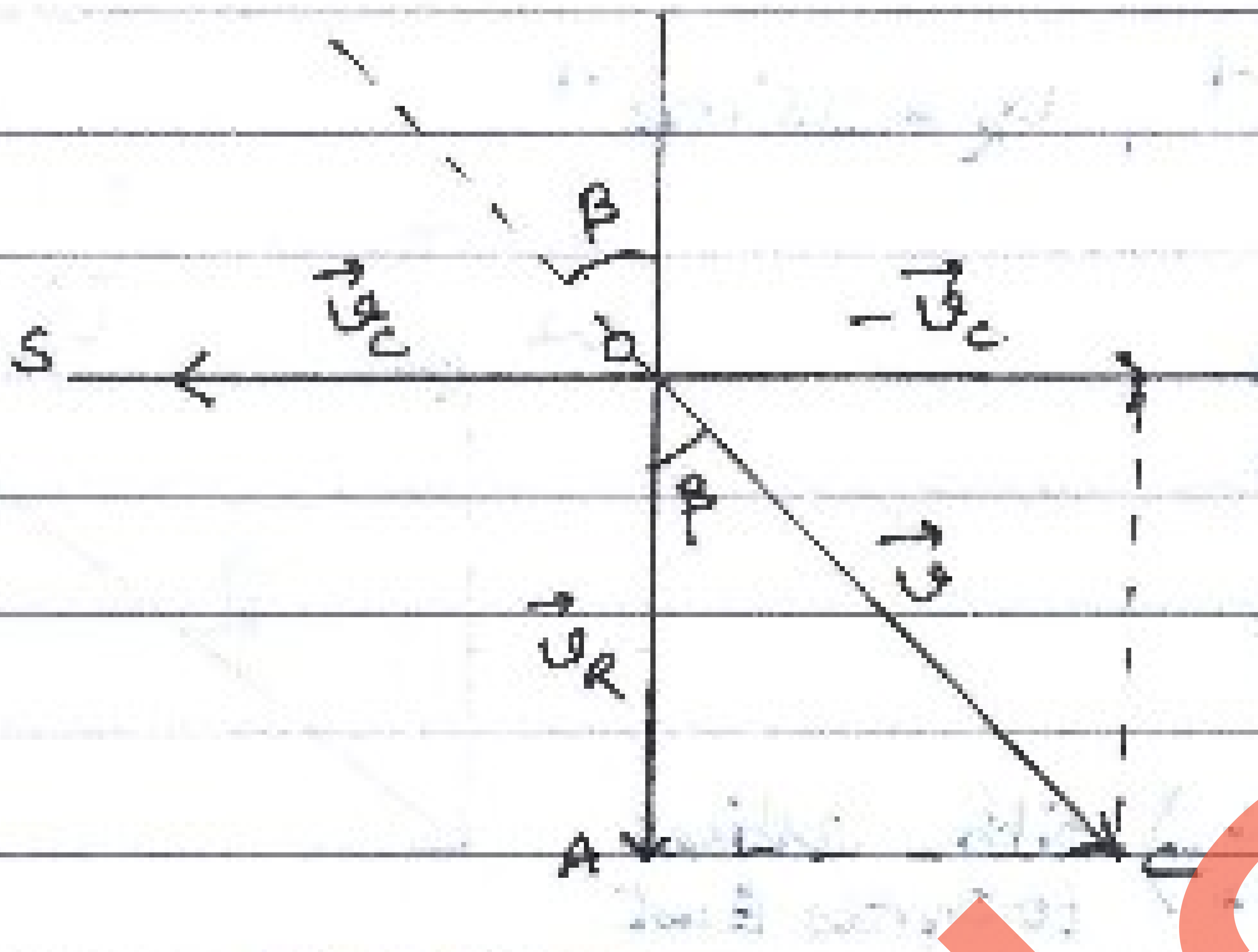
If θ is the angle which $\vec{v}_{RM}$ makes with the vertical direction then,

$$\tan\theta = \frac{v_M}{v_R}$$

$$\theta = \tan^{-1}\left(\frac{v_M}{v_R}\right)$$ towards west

	Numerical on velocity of man w.r.t rain
1.	Rain is falling vertically with a speed of 25 ms-1. A cyclist is going at a speed of 10ms⁻ in the north to south direction. What is the apparent speed and direction of the rain?
2.	Rain is falling vertically with a speed of $35ms^{-1}$. A woman rides a bicycle with a speed of $12ms^{-1}$ in East to West direction. What is the direction in which she should hold her umbrella?
3.	Rain is falling vertically with a speed of $35ms^{-1}$. Winds starts blowing after sometime with a speed of 12ms-1 in East to West direction. In which direction should a boy waiting at a bus stop hold his umbrella?

Ans - 1



$$v = \sqrt{v_R^2 + v_c^2}$$

$$= \sqrt{25^2 + 10^2}$$

$$= \sqrt{725} = 26.9 \text{ km h}^{-1}$$

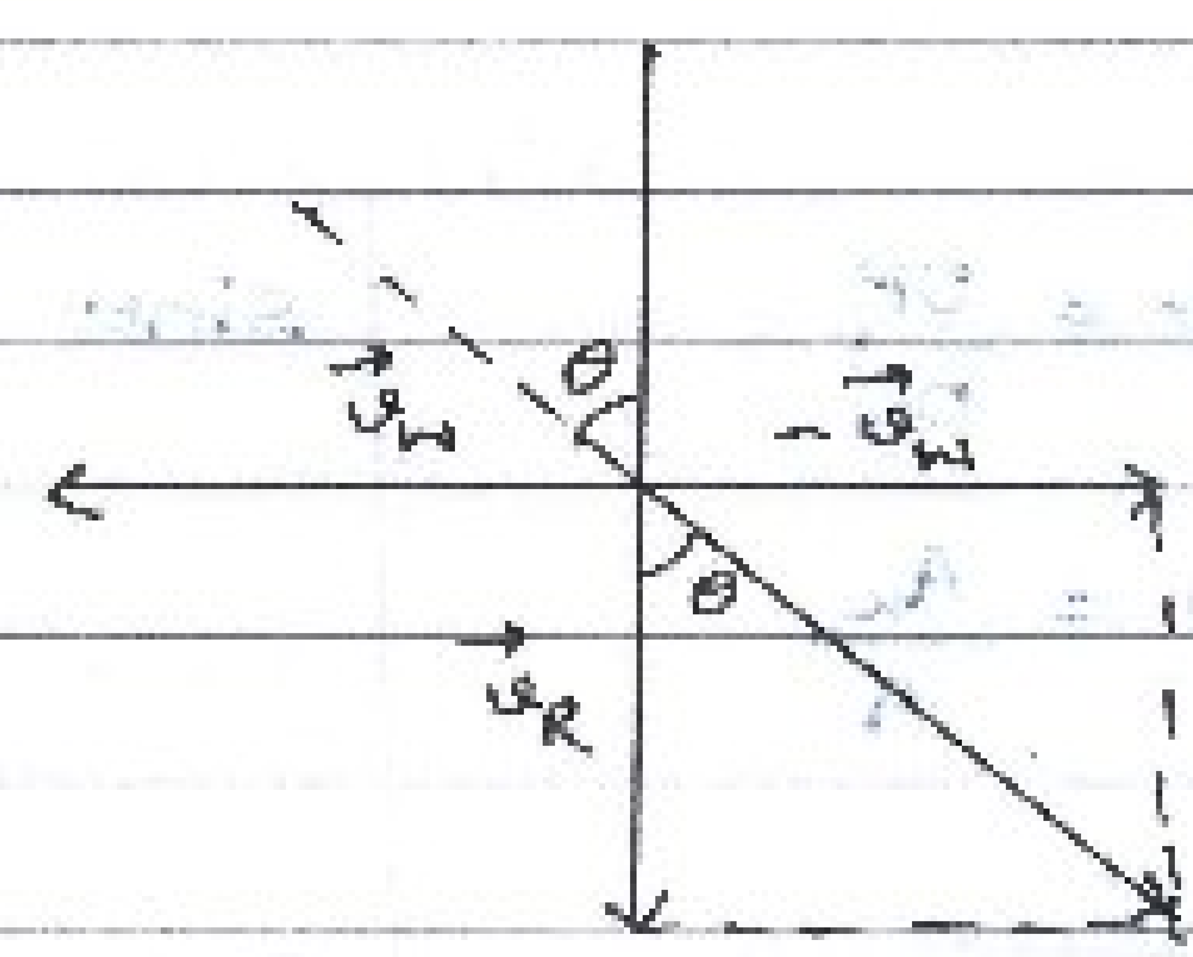
If β is the angle $\vec{v}$ makes with vertical, then

$$\tan\beta = \frac{AC}{OA}$$

$$= \frac{10}{25}$$

$$\beta = \tan^{-1}\left(\frac{10}{25}\right)$$

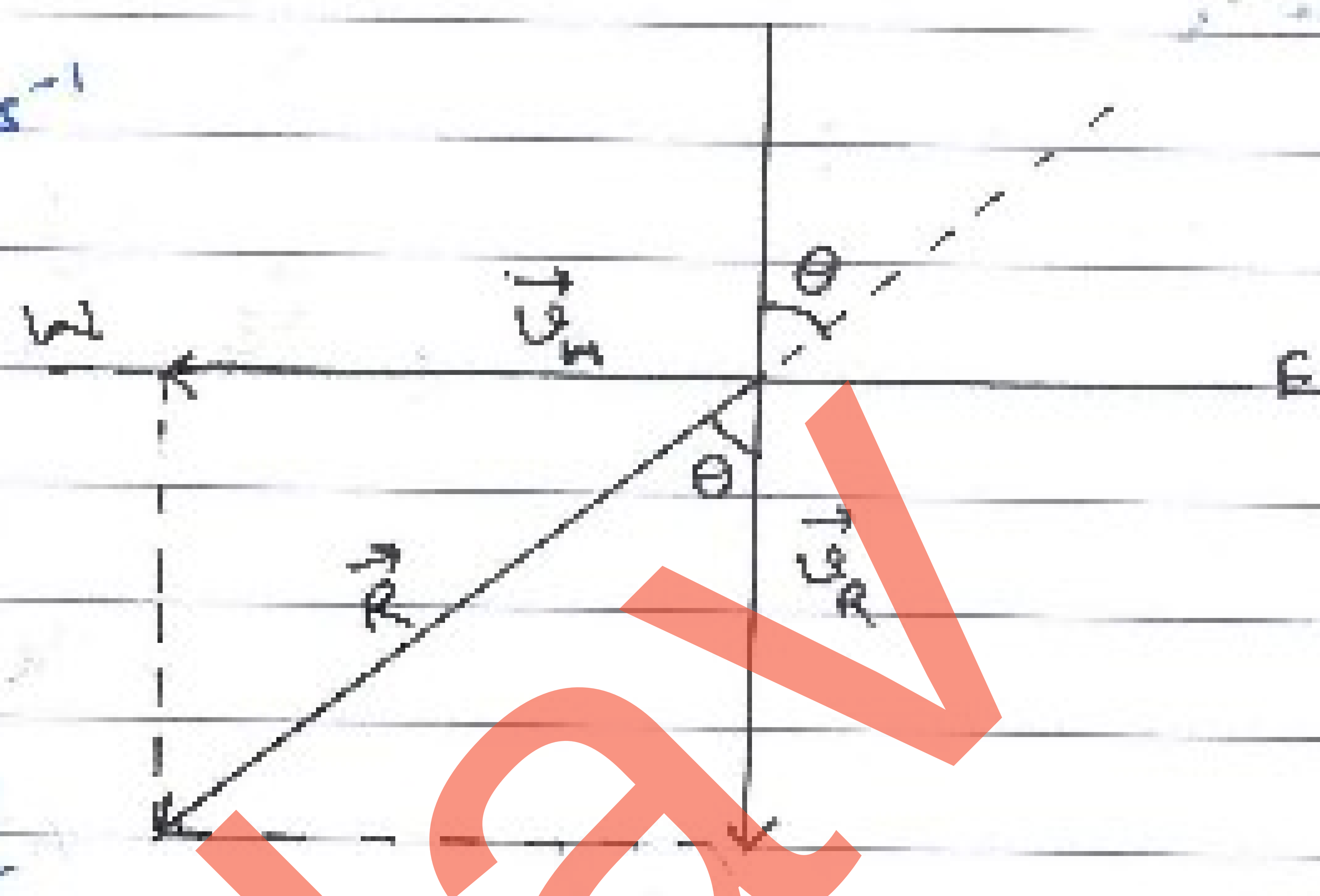
Ans - 2



$$\tan\theta = \frac{v_w}{v_R} = \frac{12}{35}$$

$\theta = \tan^{-1}\left(\frac{12}{35}\right)$ with vertical towards west

Ans-3 $v_w = 12\ ms^{-1}$, $v_R = 35\ ms^{-1}$

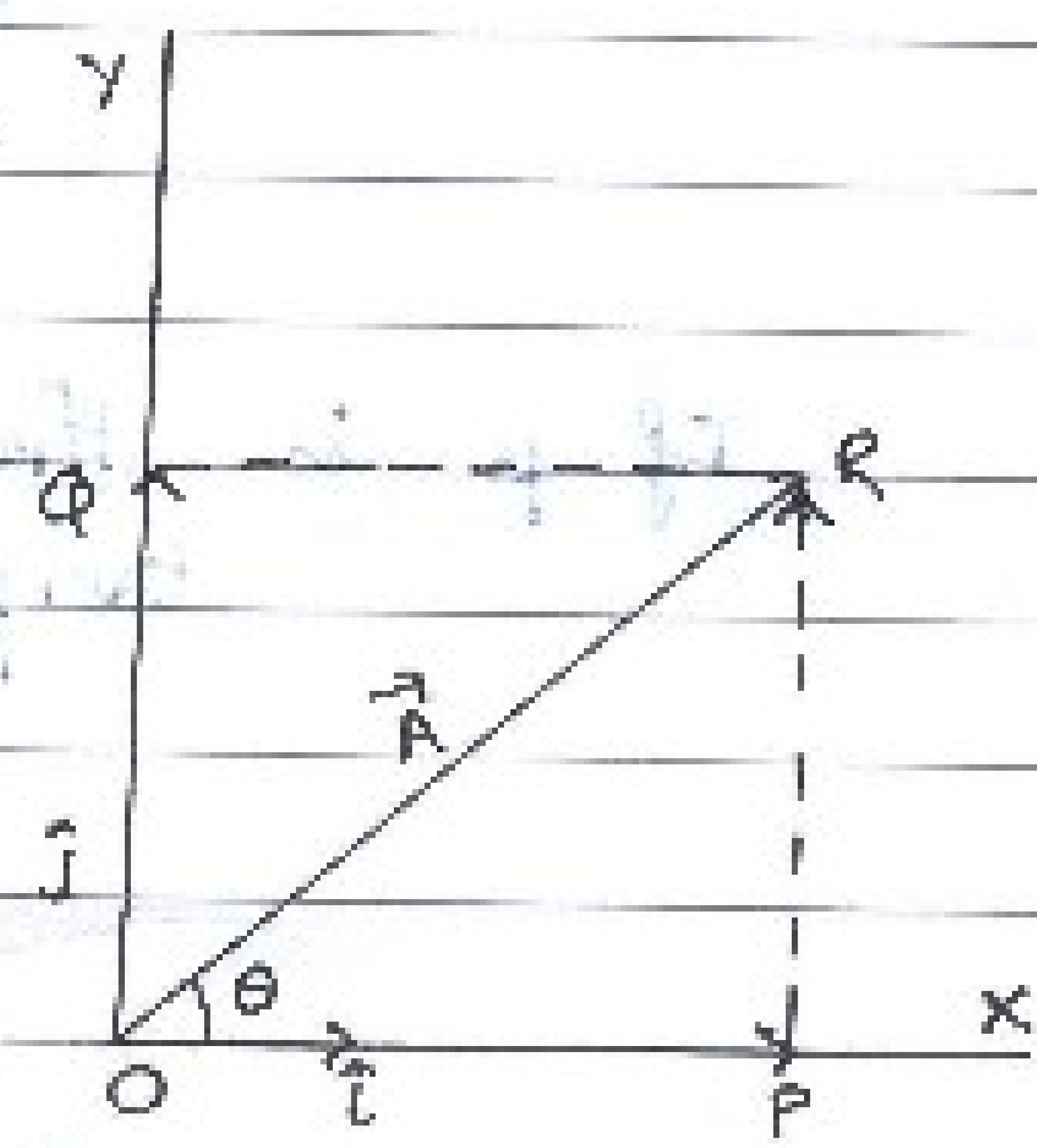


$\tan\theta = \frac{v_w}{v_R}$

$= \frac{12}{35}$

$\theta = \tan^{-1}\left(\frac{12}{35}\right)$ with vertical towards East

Rectangular components of a vector in a plane
(Resolution of vectors)



$\vec{A}(\vec{OR}) \to \vec{OP}$, $\vec{OQ}$ { Rectangular components of vector $\vec{A}$

from triangle law

$\vec{OR} = \vec{OP} + \vec{PR}$

$\vec{A} = \vec{A_x} + \vec{A_y}$

$= A_x\hat{i} + A_y\hat{j}$

In rt. ΔOPR

$\cos\theta = \frac{OP}{OR}$

$\cos\theta = \frac{A_x}{A}$

$\boxed{A_x = A\cos\theta}$

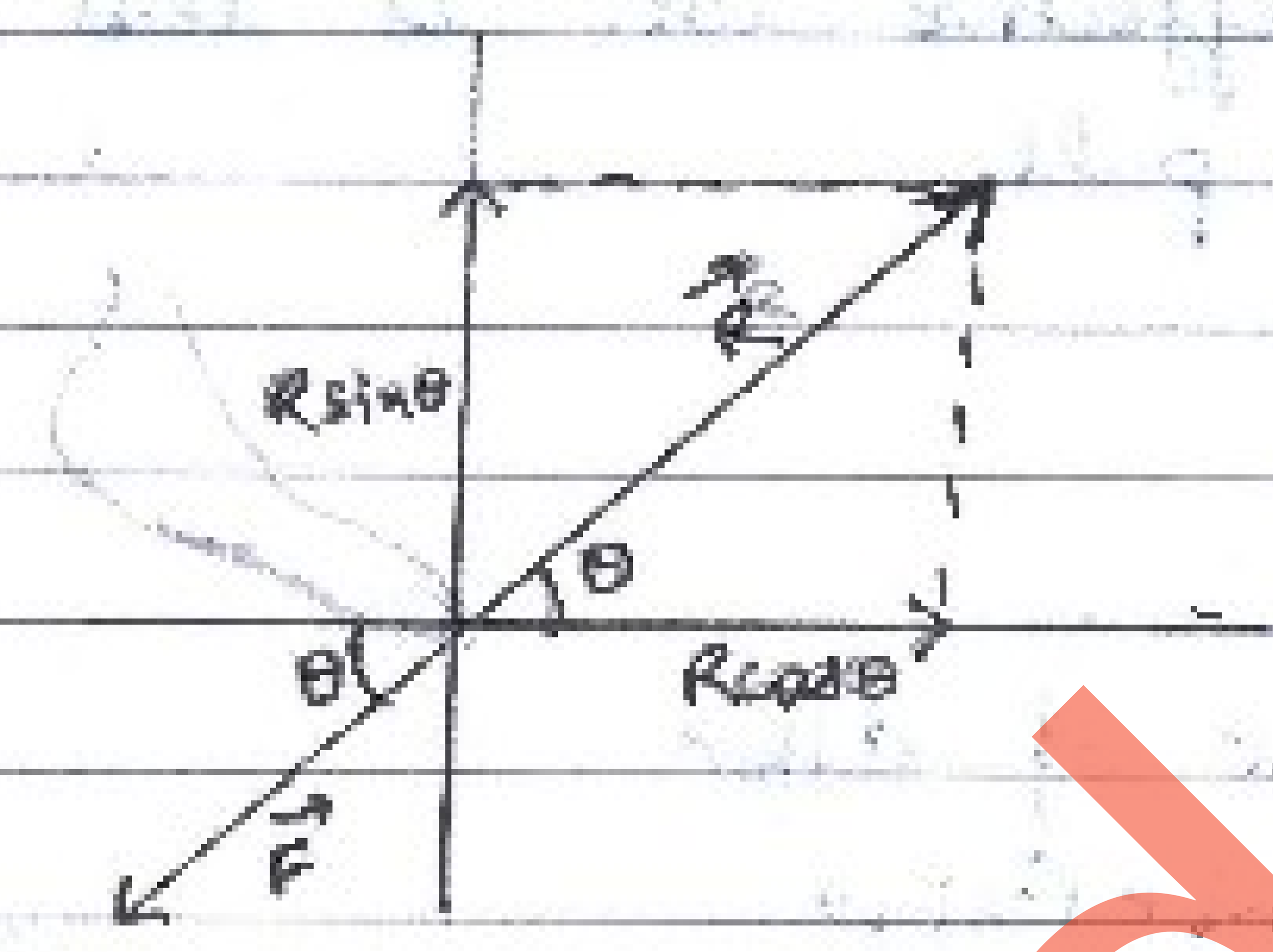
$\sin\theta = \frac{PR}{OR}$

$= \frac{A_y}{A}$

$\boxed{A_y = A\sin\theta}$

Applications of Resolution of vectors :

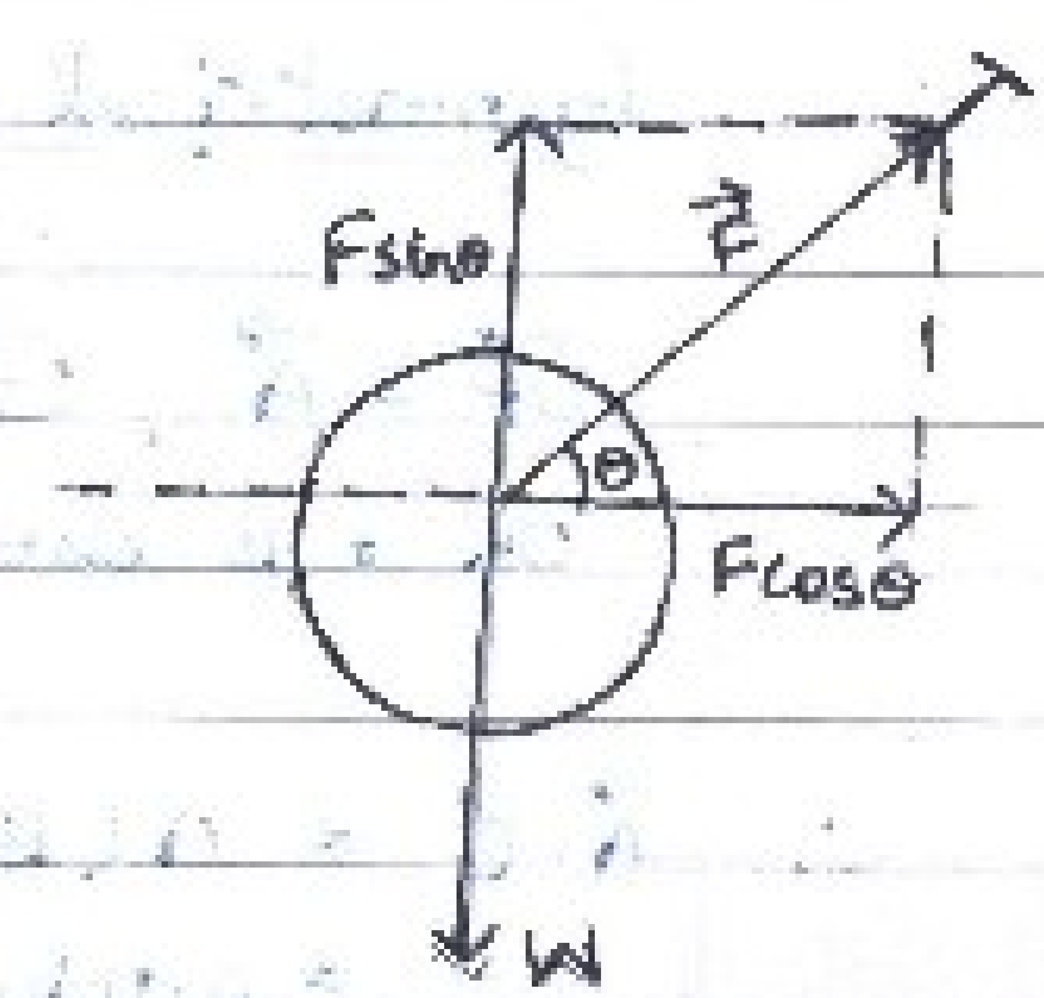
(a) Walking of a man



- While walking a person presses the ground with his feet backward by a force F at an angle θ with ground.
- In reaction the ground exerts an equal and opposite reaction R on the feet.
- R can be resolved in 2 rectangular components :
  - R sinθ – balances the body weight
  - R cosθ – helps to walk forward

(b) It is easier to pull a lawn roller than to push it.

Consider a lawn roller of weight W.
Let it be pulled or pushed by force $\vec{F}$ making an angle θ with horizontal direction.



In pulling
Effective weight of roller
$= W - F\sin\theta$

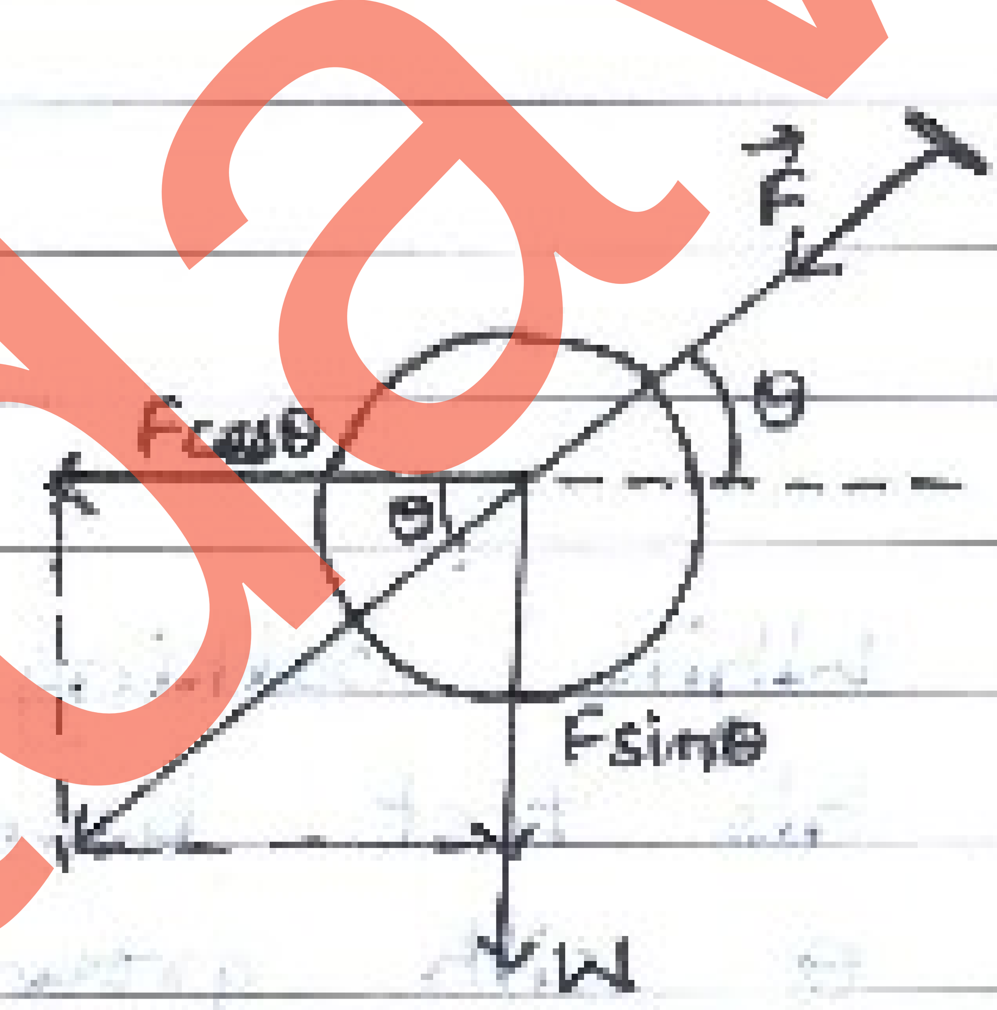
Effective horizontal pulling force
$= F\cos\theta$

As, force required to move a body is directly proportional to effective weight of body.

In pulling effective wt. is less so it is easier to pull.

<u>In pushing</u>

Effective weight of roller
$= W + F\sin\theta$



Effective horizontal pushing force
$= F\cos\theta$

In pushing effective wt. is more so it is difficult to push.

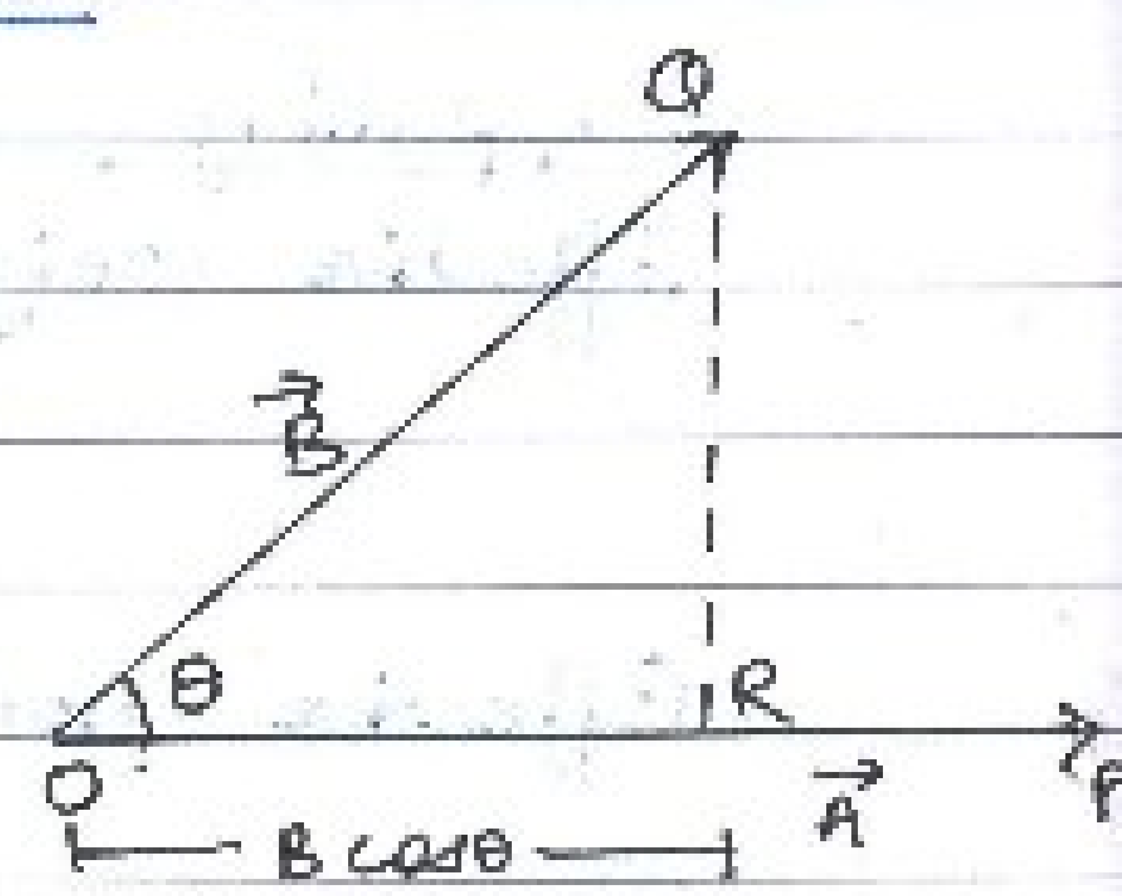
<u>Dot/Scalar Product of 2 vectors</u>

$$\boxed{\vec{A}\cdot\vec{B} = AB\cos\theta}$$

where A, B – magnitudes of vectors $\vec{A}$ & $\vec{B}$ resp.
θ – angle betn $\vec{A}$ & $\vec{B}$

<u>Geometrical intepretation of dot product</u>

$\vec{OP} = \vec{A}$, $\vec{OQ} = \vec{B}$, $\angle POQ = \theta$
$OR = B\cos\theta$ (projection of $\vec{B}$ on $\vec{A}$)



$\therefore$ $\vec{A}\cdot\vec{B} = A(B\cos\theta)$
$=$ product of magnitude of $\vec{A}$ & component of $\vec{B}$ along $\vec{A}$.

Note -

1. When 2 vectors are parallel $\theta = 0^\circ$
$\vec{A}\cdot\vec{B} = AB\cos 0^\circ = AB$

2. When 2 vectors are mutually perpendicular, $\theta = 90^\circ$
$\vec{A}\cdot\vec{B} = 0$

3. When 2 vectors are anti-parallel, $\theta = 180^\circ$
$\vec{A}\cdot\vec{B} = -AB$

From 1 $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

From 2 $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

Dot-product in cartesian co-ordinates

$$\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\begin{aligned}\vec{A}\cdot\vec{B} &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \\ &= a_1b_1(\hat{i}\cdot\hat{i}) + a_1b_2(\hat{i}\cdot\hat{j}) + a_1b_3(\hat{i}\cdot\hat{k}) + a_2b_1(\hat{j}\cdot\hat{i}) \\ &\quad + a_2b_2(\hat{j}\cdot\hat{j}) + a_2b_3(\hat{j}\cdot\hat{k}) + a_3b_1(\hat{k}\cdot\hat{i}) + a_3b_2(\hat{k}\cdot\hat{j}) \\ &\quad + a_3b_3(\hat{k}\cdot\hat{k}) \\ &= a_1b_1 + 0 + 0 + 0 + a_2b_2 + 0 + 0 + 0 + a_3b_3\end{aligned}$$

$$\boxed{\vec{A}\cdot\vec{B} = a_1b_1 + a_2b_2 + a_3b_3}$$

Properties of dot product

1. $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$
2. $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$
3. $\boxed{\vec{A}\cdot\vec{A} = AA\cos 0^\circ = A^2}$

Numerical on Scalar (Dot) Product

1. If $\vec{R}=\vec{A}-\vec{B}$, show that $R^2=A^2+B^2-2AB\cos\theta$ where θ is the angle between $\vec{A}$ and $\vec{B}$.
2. Determine the value of n, so that vectors $\vec{A}=\hat{i}+5\hat{j}+n\hat{k}$ and $\vec{B}=2\hat{i}-\hat{j}+\hat{k}$ are perpendicular. [n=3]
3. Find the angle between the vectors $\vec{A}=\hat{i}-2\hat{j}-5\hat{k}$ and $\vec{B}=2\hat{i}+\hat{j}-4\hat{k}$ $\left[\theta=\cos^{-1}\left(\frac{20}{\sqrt{30}\times\sqrt{21}}\right)\right]$
4. If $|\vec{A}+\vec{B}|=|\vec{A}-\vec{B}|$, find the angle between $\vec{A}$ and $\vec{B}$. [90°]
5. Find the magnitude and direction of (i) $\hat{i}+\hat{j}$ [$\sqrt{2}$, 45°]
6. Find the components of vector $\vec{A}=2\hat{i}+3\hat{j}$ along the directions of vectors $\hat{i}+\hat{j}$ $\left[\frac{5}{2}(\hat{i}+\hat{j})\right]$
7. Prove $|\vec{A}+\vec{B}|^2-|\vec{A}-\vec{B}|^2=4\vec{A}.\vec{B}$
8. If $\vec{A}$ and $\vec{B}$ are perpendicular to each other, prove that $|\vec{A}+\vec{B}|=\sqrt{A^2+B^2}$.
9. Show that the vectors $\vec{A}=\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{B}=2\hat{i}-\hat{j}$ are perpendicular to each other.
10. If unit vectors $\hat{A}$ and $\hat{B}$ are inclined at an angle θ, then prove that $|\hat{A}+\hat{B}|=2\sin\frac{\theta}{2}$.

Ans - 1 $\vec{R}=\vec{A}-\vec{B}$

$\vec{R}.\vec{R}=(\vec{A}-\vec{B}).(\vec{A}-\vec{B})$

$=\vec{A}.\vec{A}-\vec{A}.\vec{B}-\vec{B}.\vec{A}+\vec{B}.\vec{B}$

$R^2=A^2-AB\cos\theta-BA\cos\theta+B^2$

$R^2=A^2+B^2-2AB\cos\theta$

Ans - 2 $\vec{A}=\hat{i}+5\hat{j}+n\hat{k}$

$\vec{B}=2\hat{i}-\hat{j}+\hat{k}$

$\vec{A}.\vec{B}=(\hat{i}+5\hat{j}+n\hat{k}).(2\hat{i}-\hat{j}+\hat{k})$

$=2-5+n$

ATQ $\vec{A}.\vec{B}=0$ $[\because \vec{A}\ \&\ \vec{B}$ are $\perp^r]$

$n-3=0$

$n=3$

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Ans-3 $\vec{A} = \hat{i} - 2\hat{j} - 5\hat{k}$
$\vec{B} = 2\hat{i} + \hat{j} - 4\hat{k}$

$A = \sqrt{(1)^2 + (-2)^2 + (-5)^2} = \sqrt{30}$

$B = \sqrt{(2)^2 + (1)^2 + (-4)^2} = \sqrt{21}$

$\vec{A}\cdot\vec{B} = (\hat{i} - 2\hat{j} - 5\hat{k})\cdot(2\hat{i} + \hat{j} - 4\hat{k})$
$= 2 - 2 + 20$
$= 20$

$\vec{A}\cdot\vec{B} = AB\cos\theta$
$20 = \sqrt{30}\cdot\sqrt{21}\cos\theta$

$\theta = \cos^{-1}\left(\frac{20}{\sqrt{30}\sqrt{21}}\right)$

Ans-4 $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$
$|\vec{A} + \vec{B}|^2 = |\vec{A} - \vec{B}|^2$
$(\vec{A} + \vec{B})\cdot(\vec{A} + \vec{B}) = (\vec{A} - \vec{B})\cdot(\vec{A} - \vec{B})$
$A^2 + B^2 + 2AB\cos\theta = A^2 + B^2 - 2AB\cos\theta$
$4AB\cos\theta = 0$
$\cos\theta = 0$
$\theta = 90^\circ$

Ans-5 $|\hat{i} + \hat{j}| = \sqrt{1^2 + 1^2} = \sqrt{2}$

Direction of $(\hat{i} + \hat{j})$ along $(\hat{i})$ or $\hat{j}$
$(\hat{i} + \hat{j})\cdot\hat{i} = |\hat{i} + \hat{j}||\hat{i}|\cos\theta$
$\cancel{\sqrt{2}}\ 1 = \sqrt{2}\cos\theta$
$\cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45^\circ$

Ans-6 Unit vector along $(\hat{i}+\hat{j})$ is

$$\hat{a} = \frac{\hat{i}+\hat{j}}{\sqrt{1^2+1^2}} = \frac{\hat{i}+\hat{j}}{\sqrt{2}}$$

Magnitude of $\vec{A}$ along $(\hat{i}+\hat{j})$ is

$$\vec{A}\cdot\hat{a} = (2\hat{i}+3\hat{j})\cdot\frac{(\hat{i}+\hat{j})}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}(2+3)$$

$$= \frac{5}{\sqrt{2}}$$

Component of $\vec{A}$ along $(\hat{i}+\hat{j})$

$$(\vec{A}\cdot\hat{a})\hat{a} = \frac{5}{\sqrt{2}}\frac{(\hat{i}+\hat{j})}{\sqrt{2}} = \frac{5}{2}(\hat{i}+\hat{j})$$

Ans-7 L.H.S $= |\vec{A}+\vec{B}|^2 - |\vec{A}-\vec{B}|^2$

$$= (\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B}) - (\vec{A}-\vec{B})\cdot(\vec{A}-\vec{B})$$

$$= A^2+B^2+2\vec{A}\cdot\vec{B} - (A^2+B^2-2\vec{A}\cdot\vec{B})$$

$$= A^2+B^2+2\vec{A}\cdot\vec{B} - A^2 - B^2 + 2\vec{A}\cdot\vec{B}$$

$$= 4\,\vec{A}\cdot\vec{B}$$

R.H.S $= 4\vec{A}\cdot\vec{B}$

$\therefore$ L.H.S = R.H.S

Ans-8 $|\vec{A}+\vec{B}|^2 = (\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B})$

$$= A^2+B^2+2AB\cos\theta$$

$$= A^2+B^2+2AB\cos 90^\circ \qquad [\vec{A}\perp\vec{B}]$$

$$= A^2+B^2$$

$\therefore |\vec{A}+\vec{B}| = \sqrt{A^2+B^2}$

Ans-9

$$\vec{A} = \hat{i} + 2\hat{j} + 3\hat{k}$$
$$\vec{B} = 2\hat{i} - \hat{j}$$

$$\vec{A}\cdot\vec{B} = (\hat{i} + 2\hat{j} + 3\hat{k}).(2\hat{i} - \hat{j})$$
$$\Rightarrow 2 - 2$$
$$= 0$$

$$\vec{A}\cdot\vec{B} = AB\cos\theta$$
$$0 = AB\cos\theta$$
$$\cos\theta = 0$$
$$\theta = 90^\circ$$

Ans-10

$$|\hat{A} - \hat{B}|^2 = (\hat{A} - \hat{B}).(\hat{A} - \hat{B})$$
$$= \hat{A}.\hat{A} - \hat{A}.\hat{B} - \hat{B}.\hat{A} + \hat{B}.\hat{B}$$
$$= 1 - (1)(1)\cos\theta - (1)(1)\cos\theta + 1$$
$$= 2 - 2\cos\theta$$
$$= 2(1 - \cos\theta)$$
$$= 2\left[1 - \left(1 - 2\sin^2\frac{\theta}{2}\right)\right] \qquad \left[\because \cos 2\theta = 1 - 2\sin^2\theta\right]$$
$$= 2\left[2\sin^2\frac{\theta}{2}\right]$$
$$|\hat{A} - \hat{B}|^2 = 4\sin^2\frac{\theta}{2}$$

$$|\hat{A} - \hat{B}| = 2\sin\frac{\theta}{2}$$

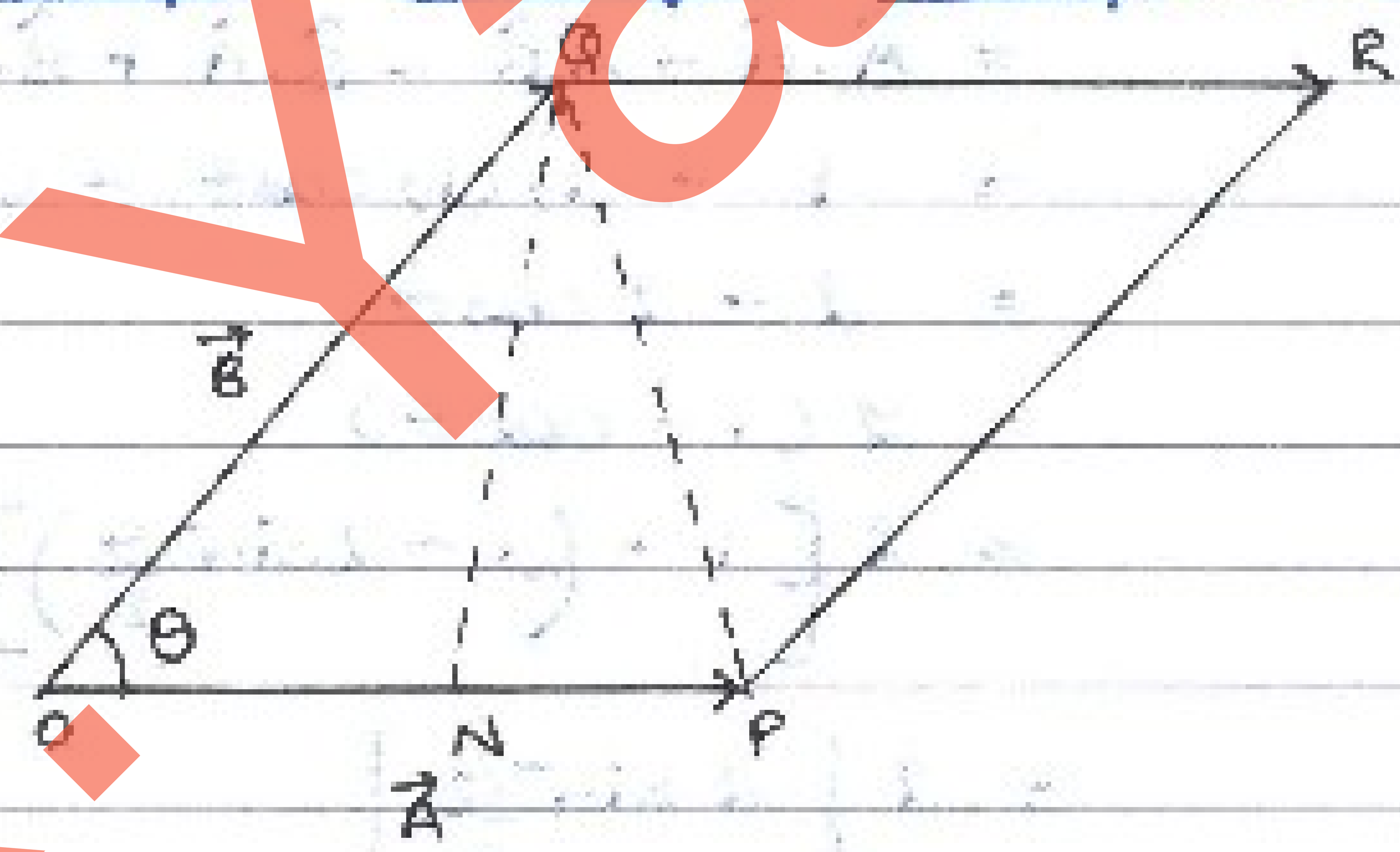
Cross/Vector product of 2 vectors

The vector product of 2 vectors $\vec{A}$ & $\vec{B}$ is another vector $\vec{C}$, whose magnitude is equal to the product of the magnitudes of the 2 vectors & sine of the smaller angle betn them.

$$\boxed{\vec{A} \times \vec{B} = \vec{C} = AB\sin\theta\,\hat{C}}$$

where $\hat{C}$ - unit vector in the direction of $\vec{C}$.
→ direction of $\hat{C}$ is $\perp^r$ to the plane containing $\vec{A}$ & $\vec{B}$.

Geometrical interpretation of vector product



$$|\vec{A} \times \vec{B}| = AB\sin\theta$$
$$= (OP)\,(OQ\sin\theta) \quad —(1)$$

In ΔONQ, $\sin\theta = \frac{NQ}{OQ}$

$$NQ = OQ\sin\theta$$

$$\boxed{|\vec{A} \times \vec{B}| = (OP)(NQ) = \text{Area of } \|^{gm}\ OPRQ}$$
$$= 2\,\frac{(OP)(NQ)}{2}$$

$$\boxed{|\vec{A} \times \vec{B}| = 2\,(\text{Area of } \Delta OQP)}$$

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Note:

1. When vectors are parallel $\theta = 180°$ or $0°$ or antiparallel

$$|\vec{A} \times \vec{B}| = 0$$

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$$

2. When vectors are perpendicular, $\theta = 90°$

$$\vec{A} \times \vec{B} = AB\,\hat{i}$$

$$\hat{i} \times \hat{j} = \hat{k} \;;\; \hat{j} \times \hat{k} = \hat{i} \;;\; \hat{k} \times \hat{i} = \hat{j}$$

Properties of cross product

① $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

② $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

③ $(\vec{A} + \vec{B}) \times (\vec{C} + \vec{D}) = \vec{A} \times \vec{C} + \vec{A} \times \vec{D} + \vec{B} \times \vec{C} + \vec{B} \times \vec{D}$

Cross product in cartesian co-ordinates:

$$\vec{A} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\vec{B} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \hat{i}(a_2b_3 - a_3b_2) - \hat{j}(a_1b_3 - a_3b_1) + \hat{k}(a_1b_2 - a_2b_1)$$

Numerical on cross product

Q1. Prove that

1. $|\vec{A}\times\vec{B}|^2 + (\vec{A}.\vec{B})^2 = A^2B^2$
2. $(\vec{A}-\vec{B})\times(\vec{A}+\vec{B}) = 2(\vec{A}\times\vec{B})$

Q2. Show that the given vectors are parallel

1. $\vec{A} = 2\hat{i} - 3\hat{j} - \hat{k}$ and $\vec{B} = -6\hat{i} - 9\hat{j} + 3\hat{k}$
2. $\vec{A} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{B} = 2\hat{i} + 4\hat{j} + 6\hat{k}$

Q3. Given that $\vec{A}\times\vec{B} = \vec{B}\times\vec{C} = \vec{0}$. If $\vec{A}, \vec{B}$ and $\vec{C}$ are not null vectors, find the value of $\vec{C}\times\vec{A}$

Q4. Find a unit vector perpendicular to each of the vectors $\vec{A} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{B} = 2\hat{i} - 2\hat{j} + 4\hat{k}$ and the angle between them.

Q5. A vector $\vec{A}$ has magnitude 6 units and is directed along positive X-axis; another vector $\vec{B}$ has magnitude 4 units and lies in the X-Y plane, making an angle of 30° with positive X-axis and an angle of 60° with positive Y-axis. Find $\vec{A}\times\vec{B}$

Ans-1

1. L.H.S $= |\vec{A}\times\vec{B}|^2 + (\vec{A}.\vec{B})^2$

$$= (AB\sin\theta)^2 + (AB\cos\theta)^2$$
$$= A^2B^2\sin^2\theta + A^2B^2\cos^2\theta$$
$$= A^2B^2(\sin^2\theta + \cos^2\theta)$$
$$= A^2B^2$$

R.H.S $= A^2B^2$

L.H.S = R.H.S

2. L.H.S $= (\vec{A}-\vec{B})\times(\vec{A}+\vec{B})$

$$= \vec{A}\times\vec{A} + \vec{A}\times\vec{B} - \vec{B}\times\vec{A} - \vec{B}\times\vec{B}$$
$$= 0 + \vec{A}\times\vec{B} - (-\vec{A}\times\vec{B}) - 0$$
$$= \vec{A}\times\vec{B} + \vec{A}\times\vec{B}$$
$$= 2(\vec{A}\times\vec{B})$$

Ans-2

1. $\vec{A} = 2\hat{i} - 3\hat{j} - \hat{k}$

$\vec{B} = -6\hat{i} + 9\hat{j} + 3\hat{k}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & -1 \\ -6 & 9 & 3 \end{vmatrix}$$

$$= \hat{i}(-9+9) - \hat{j}(6-6) + \hat{k}(18-18)$$
$$= \vec{0}$$

$$\vec{A} \times \vec{B} = AB\sin\theta$$
$$0 = AB\sin\theta$$
$$\sin\theta = 0$$
$$\theta = 0^\circ$$
$\therefore \vec{A} \parallel \vec{B}$

2. $\vec{A} = \hat{i} + 2\hat{j} + 3\hat{k}$
$\vec{B} = 2\hat{i} + 4\hat{j} + 6\hat{k}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{vmatrix}$$

$$= \hat{i}(12-12) - \hat{j}(6-6) + \hat{k}(4-4)$$
$$= \vec{0}$$

$$\vec{A} \times \vec{B} = AB\sin\theta$$
$$0 = AB\sin\theta$$
$$\therefore \sin\theta = 0$$
$$\therefore \theta = 0^\circ$$

$\therefore \vec{A} \parallel \vec{B}$

Ans-3 $\vec{A} \times \vec{B} = 0$

i.e. $\vec{A} \parallel \vec{B}$ —— (1)

Also $\vec{B} \times \vec{C} = 0$

i.e. $\vec{B} \parallel \vec{C}$ —— (2)

from (1) & (2)

$\vec{C} \parallel \vec{A}$ i.e. $\vec{C} \times \vec{A} = \vec{0}$

Ans-4 $\vec{A} = 3\hat{i} + \hat{j} + 2\hat{k}$

$\vec{B} = 2\hat{i} - 2\hat{j} + 4\hat{k}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix}$$

$$= \hat{i}(4+4) - \hat{j}(12-4) + \hat{k}(-6-2)$$

$$= 8\hat{i} - 8\hat{j} - 8\hat{k}$$

$A = \sqrt{3^2 + 1^2 + 2^2} = \sqrt{9+1+4} = \sqrt{14}$

$B = \sqrt{(2)^2 + (-2)^2 + 4^2} = \sqrt{4+4+16} = \sqrt{24}$

~~Now, $|\vec{A} \times \vec{B}| = AB \sin\theta$~~

Vector $\vec{A} \times \vec{B}$ is $\perp^r$ to both $\vec{A}$ & $\vec{B}$.

Let $\hat{n}$ be unit vector along $\vec{A} \times \vec{B}$

$$\therefore \hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$$

$|\vec{A} \times \vec{B}| = \sqrt{8^2 + (-8)^2 + (-8)^2} = 8\sqrt{3}$

$\therefore \quad \hat{n} = \dfrac{8\hat{i} - 8\hat{j} - 8\hat{k}}{8\sqrt{3}} = \dfrac{\hat{i} - \hat{j} - \hat{k}}{\sqrt{3}}$

Also, $\quad |\vec{A} \times \vec{B}| = AB\sin\theta$

$8\sqrt{3} = \sqrt{14}.\sqrt{24}\ \sin\theta$

$\sin\theta = \dfrac{8\sqrt{3}}{\sqrt{14}.\sqrt{24}}$

$\theta = \sin^{-1}\left(\dfrac{8\sqrt{3}}{\sqrt{14}.\sqrt{24}}\right)$

Ans-5 $\quad \vec{A} = 6\hat{i}$

$|\vec{B}| = 4$

$B_x = B\cos 30^\circ = 4\cos 30^\circ = 4 \times \dfrac{\sqrt{3}}{2} = 2\sqrt{3}$

$B_y = B\cos 60^\circ = 4\cos 60 = 4 \times \dfrac{1}{2} = 2$

or $\qquad$ or

$B\sin 30 \qquad 4\sin 30$

$\therefore \quad \vec{B} = 2\sqrt{3}\hat{i} + 2\hat{j}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 0 & 0 \\ 2\sqrt{3} & 2 & 0 \end{vmatrix}$$

$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(12-0)$

$= 12\hat{k}$

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Numerical on cross product

Q1. Prove that

1. $|\vec{A}\times\vec{B}|^2 + (\vec{A}.\vec{B})^2 = A^2B^2$
2. $(\vec{A}-\vec{B})\times(\vec{A}+\vec{B}) = 2(\vec{A}\times\vec{B})$

Q2. Show that the given vectors are parallel

1. $\vec{A} = 2\hat{i}-3\hat{j}-\hat{k}$ and $\vec{B} = -6\hat{i}+9\hat{j}+3\hat{k}$
2. $\vec{A} = \hat{i}+2\hat{j}+3\hat{k}$ and $\vec{B} = 2\hat{i}+4\hat{j}+6\hat{k}$

Q3. Given that $\vec{A}\times\vec{B} = \vec{B}\times\vec{C} = \vec{0}$. If $\vec{A}, \vec{B}$ and $\vec{C}$ are not null vectors, find the value of $\vec{C}\times\vec{A}$

Q4. Find a unit vector perpendicular to each of the vectors $\vec{A} = 3\hat{i}+\hat{j}-2\hat{k}$ and $\vec{B} = 2\hat{i}-2\hat{j}-4\hat{k}$ and the angle between them.

Q5. A vector $\vec{A}$ has magnitude 6 units and is directed along positive X-axis; another vector $\vec{B}$ has magnitude 4 units and lies in the X-Y plane, making an angle of 30° with positive X-axis and an angle of 60° with positive Y-axis. Find $\vec{A}\times\vec{B}$

Ans-1 1. $L.H.S = |\vec{A}\times\vec{B}|^2 + (\vec{A}.\vec{B})^2$

$= (AB\sin\theta)^2 + (AB\cos\theta)^2$

$= A^2B^2\sin^2\theta + A^2B^2\cos^2\theta$

$= A^2B^2(\sin^2\theta + \cos^2\theta)$

$= A^2B^2$

$R.H.S = A^2B^2$

$L.H.S = R.H.S$

2. $L.H.S = (\vec{A}-\vec{B})\times(\vec{A}+\vec{B})$

$= \vec{A}\times\vec{A} + \vec{A}\times\vec{B} - \vec{B}\times\vec{A} - \vec{B}\times\vec{B}$

$= 0 + \vec{A}\times\vec{B} - (-\vec{A}\times\vec{B}) - 0$

$= \vec{A}\times\vec{B} + \vec{A}\times\vec{B}$

$= 2(\vec{A}\times\vec{B})$

Ans-2 1. $\vec{A} = 2\hat{i}-3\hat{j}-\hat{k}$

$\vec{B} = -6\hat{i}+9\hat{j}+3\hat{k}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & -1 \\ -6 & 9 & 3 \end{vmatrix}$$

$$= \hat{i}(-9+9) - \hat{j}(6-6) + \hat{k}(18-18)$$

$$= \vec{0}$$

$$\vec{A} \times \vec{B} = AB\sin\theta$$

$$0 = AB\sin\theta$$

$$\sin\theta = 0$$

$$\theta = 0^\circ$$

$\therefore \vec{A} \parallel \vec{B}$

2. $\vec{A} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{B} = 2\hat{i} + 4\hat{j} + 6\hat{k}$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{vmatrix}$$

$$= \hat{i}(12-12) - \hat{j}(6-6) + \hat{k}(4-4)$$

$$= \vec{0}$$

$$\vec{A} \times \vec{B} = AB\sin\theta$$

$$0 = AB\sin\theta$$

$$\therefore \sin\theta = 0$$

$$\therefore \theta = 0^\circ$$

$\therefore \vec{A} \parallel \vec{B}$

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Ans-3 $\vec{A} \times \vec{B} = 0$

i.e. $\vec{A} \parallel \vec{B}$ —— (1)

Also $\vec{B} \times \vec{C} = 0$

i.e. $\vec{B} \parallel \vec{C}$ —— (2)

from (1) & (2)

$\vec{C} \parallel \vec{A}$ i.e. $\vec{C} \times \vec{A} = \vec{0}$.

Ans-4

$$\vec{A} = 3\hat{i} + \hat{j} + 2\hat{k}$$
$$\vec{B} = 2\hat{i} - 2\hat{j} + 4\hat{k}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 2 & -2 & 4 \end{vmatrix}$$

$$= \hat{i}(4+4) - \hat{j}(12-4) + \hat{k}(-6-2)$$
$$= 8\hat{i} - 8\hat{j} - 8\hat{k}$$

$$A = \sqrt{3^2 + 1^2 + 2^2} = \sqrt{9+1+4} = \sqrt{14}$$

$$B = \sqrt{(2)^2 + (-2)^2 + 4^2} = \sqrt{4+4+16} = \sqrt{24}$$

Vector $\vec{A} \times \vec{B}$ is $\perp^r$ to both $\vec{A}$ & $\vec{B}$.

Let $\hat{n}$ be unit vector along $\vec{A} \times \vec{B}$

$$\therefore \hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$$

$$|\vec{A} \times \vec{B}| = \sqrt{8^2 + (-8)^2 + (-8)^2} = 8\sqrt{3}$$

$$\therefore \quad \hat{n} = \frac{8\hat{i} - 8\hat{j} - 8\hat{k}}{8\sqrt{3}} = \frac{\hat{i} - \hat{j} - \hat{k}}{\sqrt{3}}$$

Also, $|\vec{A} \times \vec{B}| = AB\sin\theta$

$$8\sqrt{3} = \sqrt{14}.\sqrt{24}\sin\theta$$

$$\sin\theta = \frac{8\sqrt{3}}{\sqrt{14}.\sqrt{24}}$$

$$\theta = \sin^{-1}\left(\frac{8\sqrt{3}}{\sqrt{14}.\sqrt{24}}\right)$$

Ans-5

$$\vec{A} = 6\hat{i}$$

$$|\vec{B}| = 4$$

$$B_x = B\cos 30^\circ = 4\cos 30^\circ = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$$

$$B_y = B\cos 60^\circ = 4\cos 60 = 4 \times \frac{1}{2} = 2$$

or $\qquad$ or

$B\sin 30 \qquad 4\sin 30$

$$\therefore \quad \vec{B} = 2\sqrt{3}\,\hat{i} + 2\hat{j}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 0 & 0 \\ 2\sqrt{3} & 2 & 0 \end{vmatrix}$$

$$= \hat{i}(0-0) - \hat{j}(0-0) + \hat{k}(12-0)$$

$$= 12\hat{k}$$

Projectile

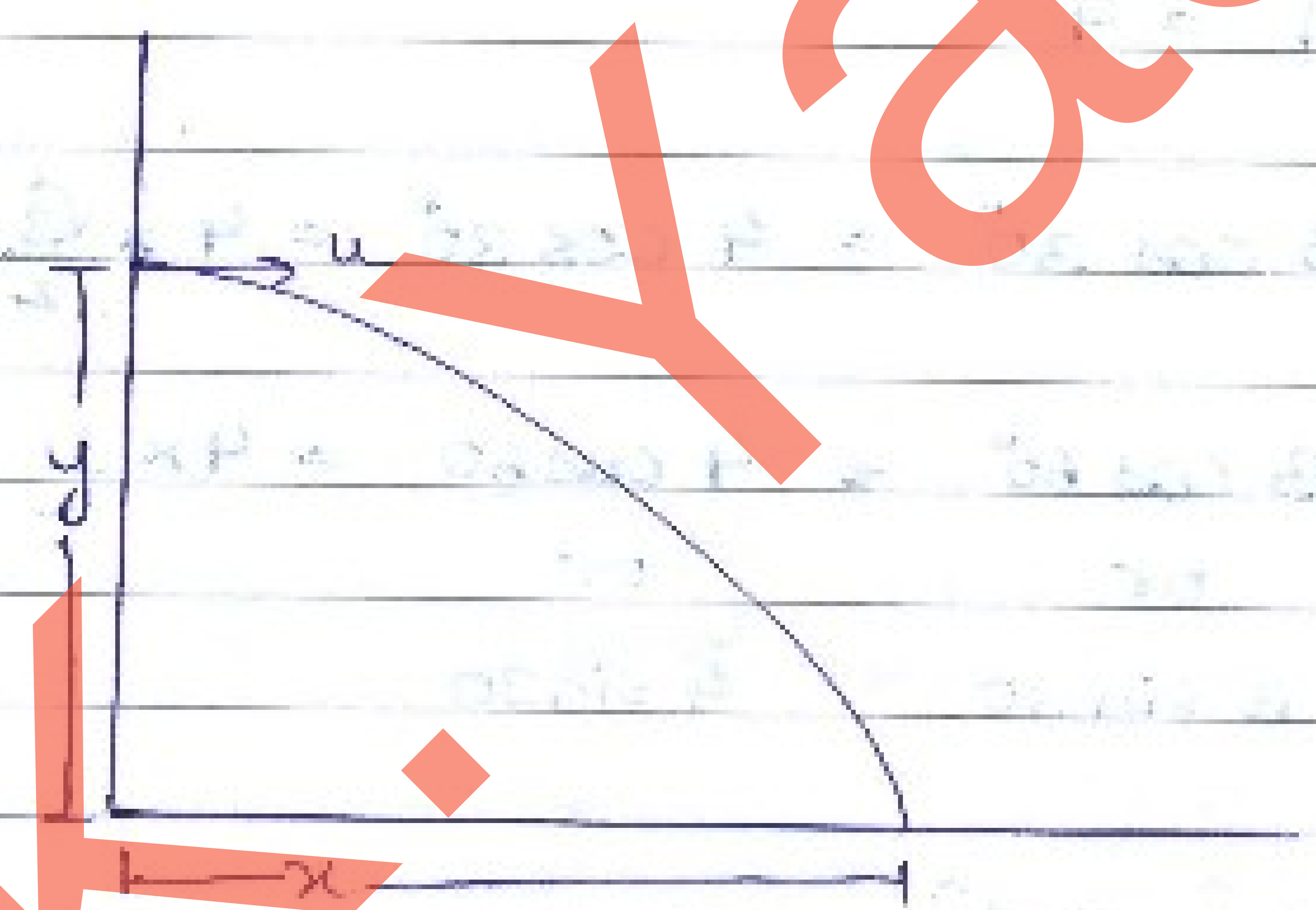
An object thrown with initial velocity and which is then allowed to move under the action of gravity alone is called projectile.

eg (i) a bomb released from a plane
(ii) a bullet shot from gun.

Trajectory - path followed by a projectile.

Horizontal projection of projectile

Consider an object thrown horizontally with an initial velocity u from a height h.



$$s_x = u_x t + \frac{1}{2} a_x t^2$$

$$x = ut + 0$$

$$t = \frac{x}{u}$$

&

$$s_y = u_y t + \frac{1}{2} a_y t^2$$

$$y = 0 + \frac{1}{2} g t^2$$

$$y = \frac{1}{2} g t^2$$

$$= \frac{1}{2} g \frac{x^2}{u^2}$$

$$y = \frac{g}{2u^2} x^2$$

$$\boxed{y = k x^2}$$

This is the eqn of parabola, so the path of a projectile is parabolic in nature.

① Time of flight (T)

$$y = \frac{1}{2} g t^2$$

$$h = \frac{1}{2} g T^2$$

$$\boxed{T = \sqrt{\frac{2h}{g}}}$$

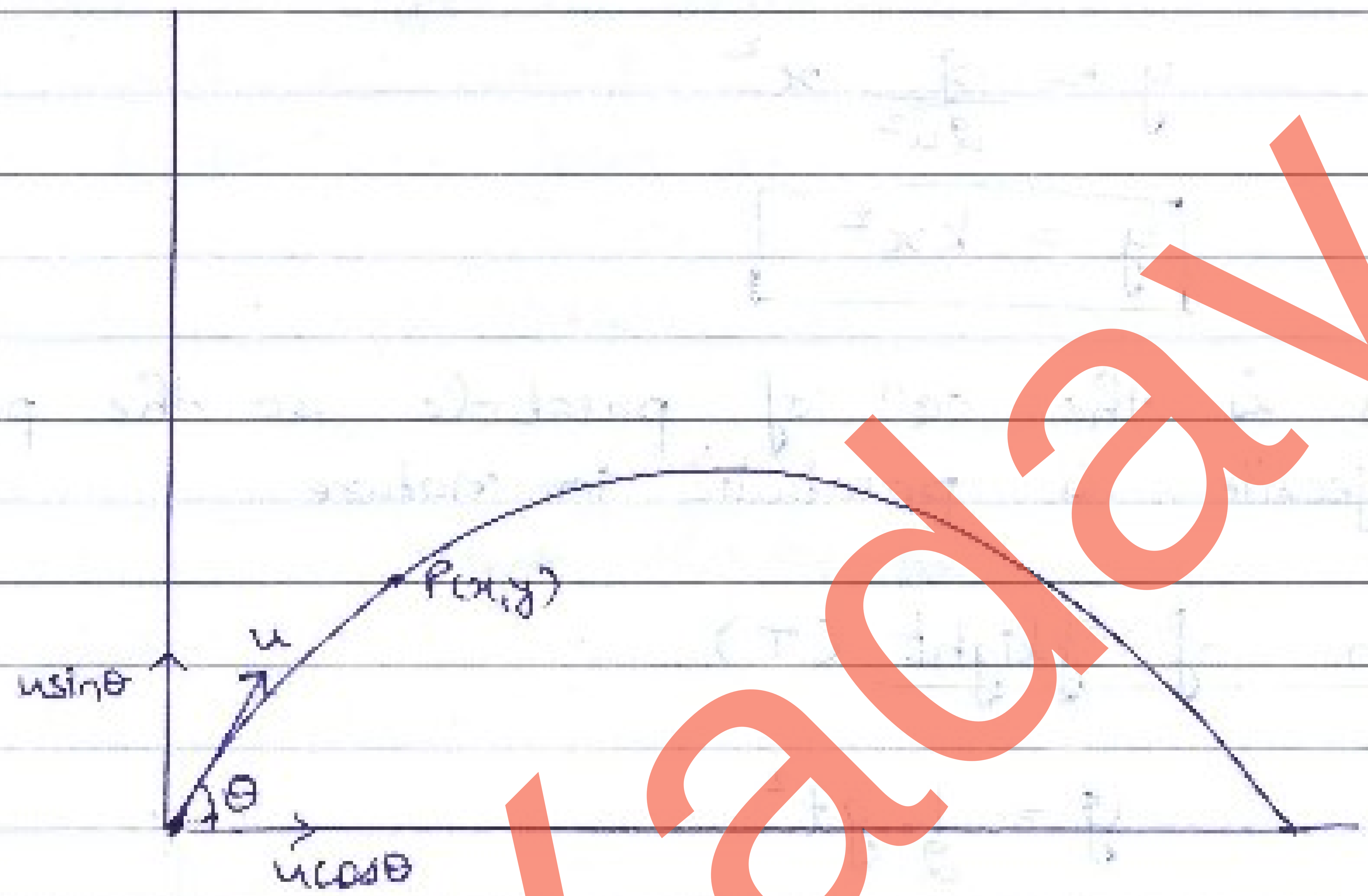
② Horizontal range (R)

$$x = ut$$

$$R = u\sqrt{\frac{2h}{g}}$$

Projectile fired at an angle with horizontal

Let a projectile be thrown with initial velocity u at an angle θ with horizontal.



Let at any instant of time t, the projectile be at pt. P(x,y)

$$S_x = u\cos\theta t + \frac{1}{2} a_x t^2$$

$$x = u\cos\theta . t$$

$$t = \frac{x}{u\cos\theta}$$

$$S_y = u_y t + \frac{1}{2} a_y t^2$$

$$y = u\sin\theta . t + \frac{1}{2}(-g) t^2$$

$$= \cancel{u}\sin\theta . \frac{x}{\cancel{u}\cos\theta} - \frac{1}{2} g \frac{x^2}{u^2\cos^2\theta}$$

$$\boxed{y = \tan\theta . x - \left(\frac{g}{2u^2\cos^2\theta}\right) x^2}$$ — eqn of parabola

① Max. height (H)

$$v_y^2 = u_y^2 + 2a_y S_y$$

$$0^2 = u^2\sin^2\theta - 2gH$$

$$\boxed{H = \frac{u^2\sin^2\theta}{2g}}$$

② Time of flight (T)

$$v_y = u_y + a_y t$$

at max. height, $v_y = 0$, $t = \frac{T}{2}$

$$\therefore \quad 0 = u\sin\theta - g\frac{T}{2}$$

$$\boxed{T = \frac{2u\sin\theta}{g}}$$

③ Horizontal range (R) – max. distance betⁿ point of projection & pt. of hitting.

$$R = u_x T$$

$$= u\cos\theta \cdot \frac{2u\sin\theta}{g}$$

$$= \frac{u^2\, 2\sin\theta\cos\theta}{g}$$

$$\boxed{R = \frac{u^2\sin 2\theta}{g}}$$

$R_{max} = \frac{u^2}{g}$ when $\sin 2\theta = 1$

$\sin 2\theta = \sin 90$

$\boxed{\theta = 45°}$

* Two angles of projection for the same range of projectile

$$R = \frac{2u^2 \sin\theta \cos\theta}{g}$$

Let the new angle is $90 - \theta$

$$\therefore \quad R' = \frac{2u^2 \sin(90-\theta)\cos(90-\theta)}{g}$$

$$= \frac{u^2\, 2\cos\theta \sin\theta}{g}$$

$$R' = \frac{u^2 \sin 2\theta}{g}$$

$$\boxed{R' = R}$$