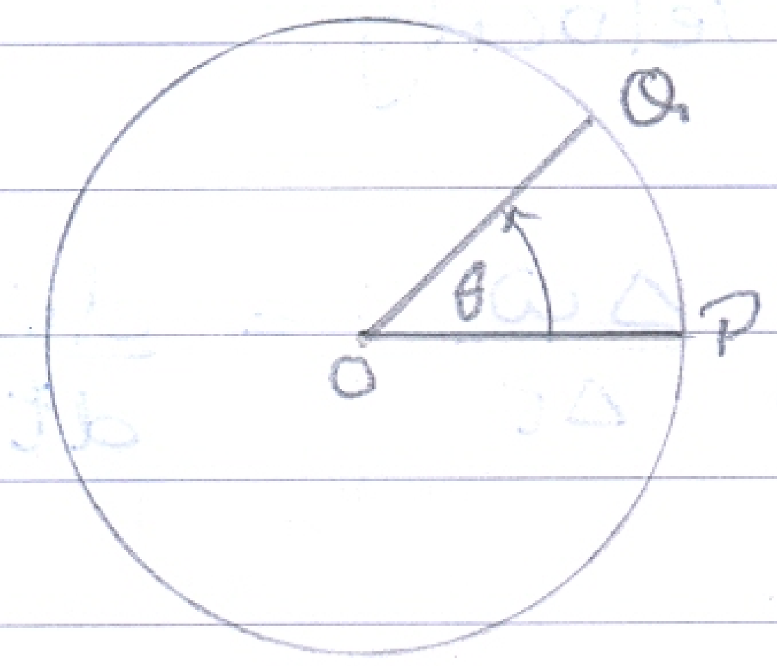
Uniform Circular motion

Angular displacement (θ)



Consider an object moving along a circular path of radius 'r' in anti-clockwise direction. Let the position of the object changes from P to Q in time 't'.

$\therefore \quad \angle POQ = \theta$

During time 't', radius vector traces out an angle θ at the axis of the circular path.

Angular displacement (θ) of the object moving around a circular path is defined as the angle traced out by the radius vector at the centre of the circular path in a given time.

$\Delta l = v \Delta t$ $\qquad [\because v = \frac{\Delta l}{\Delta t}]$

$\Delta \theta = \omega \Delta t$ $\qquad [\because \omega = \Delta\theta/\Delta t$ (by def^n)

Now, angle $= \dfrac{\text{arc}}{\text{radius}}$

$$\Delta\theta = \frac{\Delta l}{r}$$

$$\omega \Delta t = \frac{v \Delta t}{r}$$

$\boxed{\vec{v} = \omega \vec{r}}$ ← Relation bet^n linear & angular velocity

Angular acceleration (α)

It is defined as the time rate of change of angular velocity.

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt}$$

S.I. unit $\rightarrow$ $rad.s^{-2}$

Dimensional formula $\rightarrow$ $[T^{-2}]$

Relation bet^n $\vec{a}$ & α

$$v = \omega r$$

Differentiating both sides w.r.t time

$$\frac{dv}{dt} = \frac{d}{dt}(\omega r)$$

$$= \omega \frac{dr}{dt} + r \frac{d\omega}{dt}$$

$$= 0 + r \frac{d\omega}{dt}$$

$$\boxed{\vec{a} = \alpha \vec{r}}$$

Uniform circular motion

- When a point object is moving on a circular path with a constant speed, then the motion of the object is said to be a uniform circular motion.
- In uniform circular motion, the velocity of the object is changing its direction continuously so it is called uniformly accelerated motion.

Time period (T)

The time taken by the object to complete one revolution on its circular path.

Frequency (ν)

It is no. of revolutions completed by the object on its circular path in a unit time.

$$\nu = \frac{\text{No. of revolutions}}{\text{Time taken}} = \frac{1}{T}$$

Relation betn θ, ν & T

Consider a point object having a uniform circular motion with frequency ν & time period T.

When the object completes one revolution, the angle traced at its axis of circular motion is 2π.

i.e. when $t = T$

$\theta = 2\pi$ rad.

$$\therefore \boxed{\omega = \frac{\theta}{t} = \frac{2\pi}{T} = 2\pi\nu}$$

Centripetal acceleration

Acceleration acting on the object undergoing uniform circular motion is called centripetal acc. It always act on the object along the radius towards the centre of the circular path.

$$|\vec{a}| = \omega^2 r = \frac{v^2}{r}$$

Centripetal force

It is the force required to move a body uniformly in a circle.
It acts along the radius & towards the centre of the circle.

Centripetal force, F = mass x centripetal acc.

$$\boxed{F = \frac{mv^2}{r} = m\omega^2 r}$$

Centrifugal force

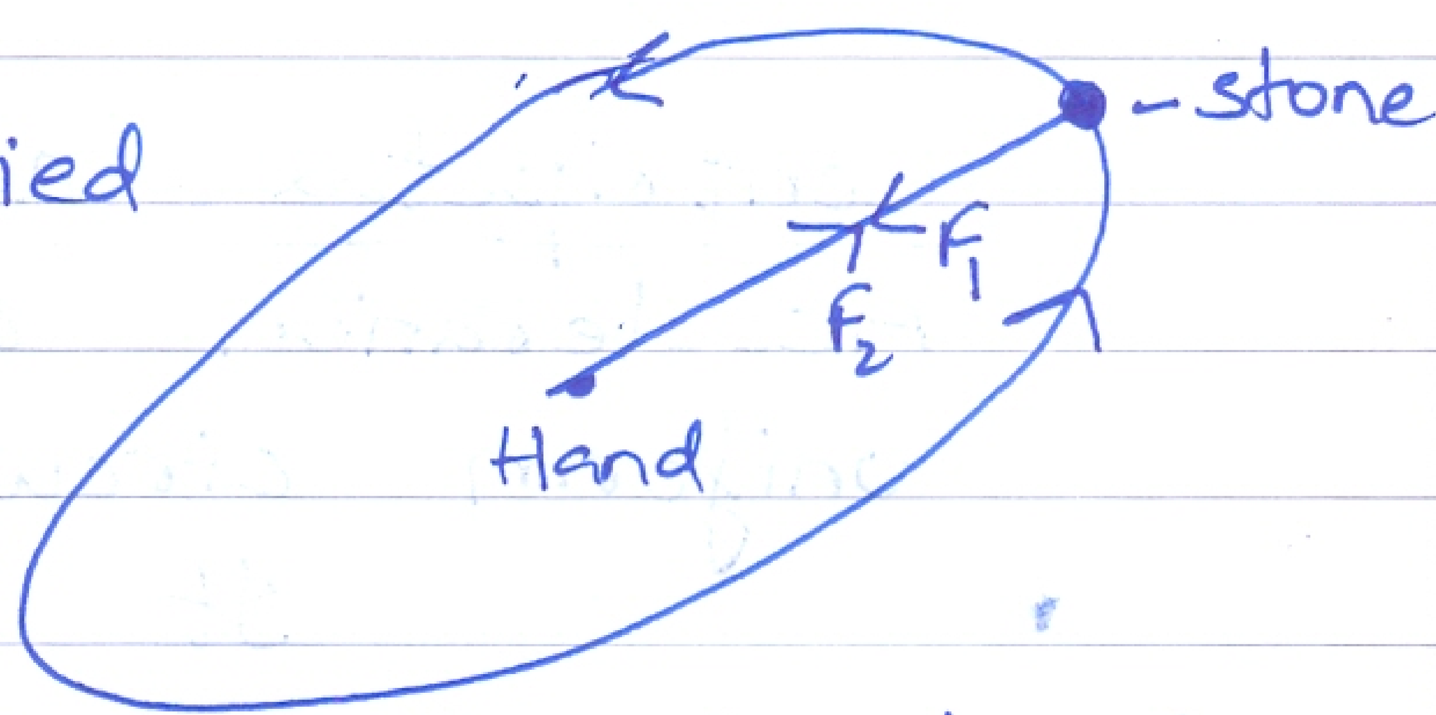
It is a force that arises when a body is moving actually along a circular path, by virtue of the tendency of the body to regain its natural straight line path.

→ It can be regarded as the reaction of centripetal force so

$$\text{centrifugal force} = \frac{mv^2}{r}$$

→ It acts along the radius & away from the centre of the circle.

Example: When a piece of stone tied to one end of string is rotated in a circle, then



F_1 (centripetal force) - applied on stone by hand
F_2 (centrifugal ") - " " hand by stone
(to regain its natural straight line path)

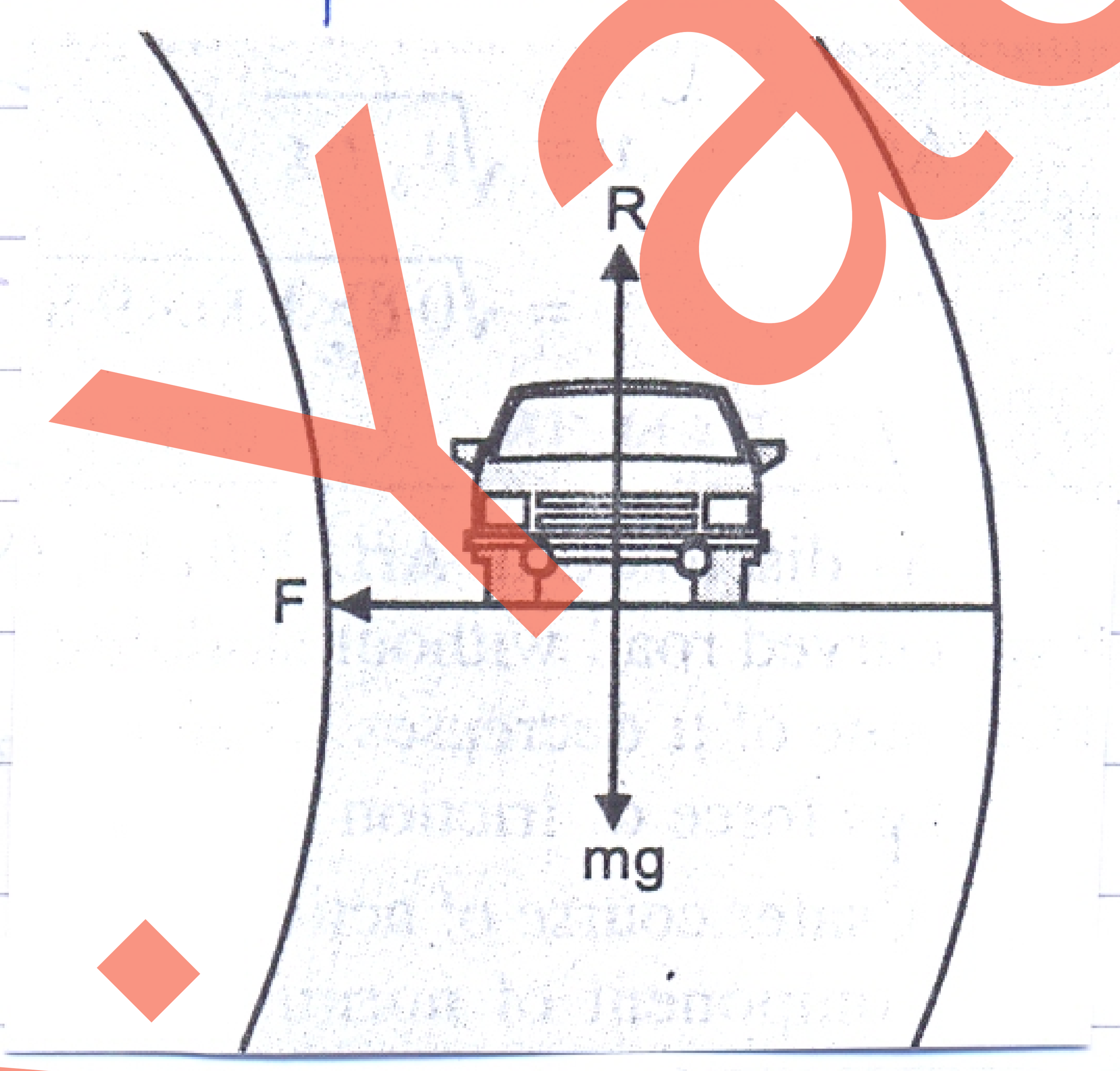
Rounding a level curved road

→ When a vehicle goes round a curved road, it requires some centripetal force.

→ While rounding the curve, the wheels of the vehicle have a tendency to leave the curved path & regain the straight line path.

→ Force of friction between the wheels & road opposes this tendency of the wheels.

→ This frictional force acts towards the centre of the circular track & provides the necessary centripetal force.



Consider a car moving on a curved road. Three forces acting on the car are

(i) Weight of car, mg - vertically downwards

(ii) Normal reaction, R - " upwards

(iii) Frictional force, F - towards the centre.

As there is no. acceleration in vertical direction

$$R = mg$$

The centripetal force is a static friction, and

$$\frac{mv^2}{r} \leq F$$

Now, $\mu_s = \frac{F}{R}$

$$F = \mu_s R = \mu_s mg$$

So, $$\frac{\cancel{m} v^2}{r} \leq \mu_s \cancel{m} g$$

$$v \leq \sqrt{\mu_s r g}$$

or, $$\boxed{v_{max} \leq \sqrt{\mu_s r g}}$$

It is the max. velocity with which the car can move without skidding

☆ v depends on (i) radius of curve
(ii) μ_s betn tyres & road
independent of mass of car.

<u>Banking of roads</u> (Basic concept)

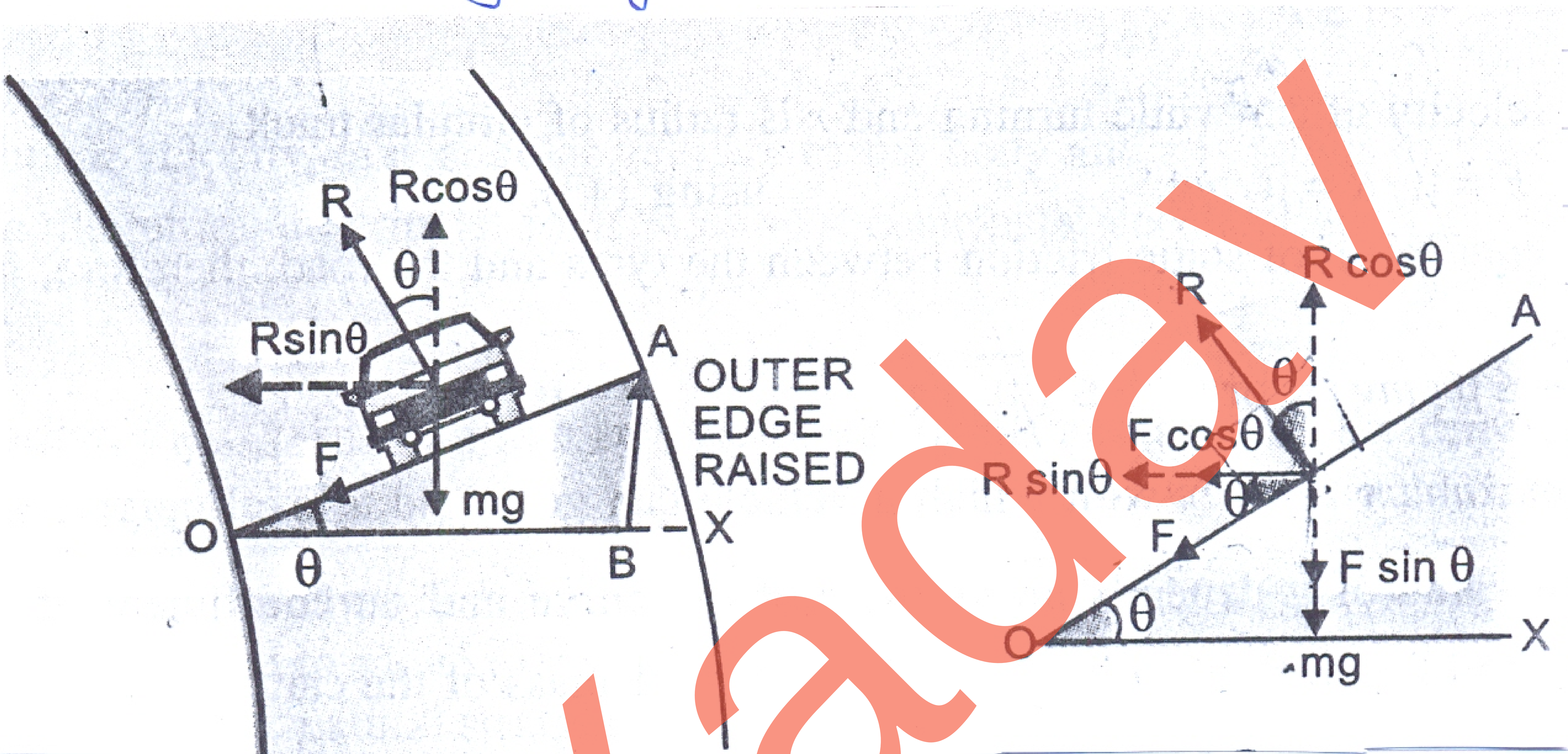
The max. permissible velocity with which a vehicle can go round a level curved road without skidding depends upon μ between the road & tyres.

The value of μ decreases
(i) when road is smooth.
(ii) tyres are worn out
(iii) road is wet

So, force of friction is not a reliable souce for providing the required centripetal force. A safer option is to raise the outer edge of the curved edge of road.

Banking of roads

The phenomenon of raising outer edge of the curved road above the inner edge is called banking of roads.



Consider a vehicle of mass 'm' moving on a curved road.

Three forces acting on the vehicle are -

(i) Weight, mg - vertically downwards

(ii) Normal reaction, R - upwards ($\perp^r$ to OA) (of banked road)

(iii) Frictional force, F - along AO. (bet^n banked road & tyres)

Resolving R into rectangular components

(i) $R\cos\theta$ → vertically upwards

(ii) $R\sin\theta$ → horizontal (towards the centre of road)

Similarly F can be resolved as $F\cos\theta$ & $F\sin\theta$

As there is no acceleration along vertical direction

$$R\cos\theta = mg + F\sin\theta \quad \text{——} (1)$$

If v - velocity of vehicle over banked circular road
r - radius of road, then

centripetal force $= \frac{mv^2}{r}$

& $R\sin\theta + F\cos\theta = \frac{mv^2}{r}$ —— (2)

But $F \leq \mu_s R$

So, eqn (1) & (2) becomes

$$R\cos\theta = mg + \mu_s R\sin\theta$$

$$R(\cos\theta - \mu_s \sin\theta) = mg$$

$$R = \frac{mg}{(\cos\theta - \mu_s\sin\theta)}$$

& $R(\sin\theta + \mu_s\cos\theta) = \frac{mv^2}{r}$

$$\frac{mg}{(\cos\theta - \mu_s\sin\theta)}(\sin\theta + \mu_s\cos\theta) = \frac{mv^2}{r}$$

$$v^2 = \frac{rg(\sin\theta + \mu_s\cos\theta)}{(\cos\theta - \mu_s\sin\theta)}$$

$$= \frac{rg\cos\theta(\tan\theta + \mu_s)}{\cos\theta(1 - \mu_s\tan\theta)}$$

$$\boxed{v_{max} = \left[\frac{rg(\mu_s + \tan\theta)}{(1 - \mu_s\tan\theta)}\right]^{1/2}}$$

It is the max. velocity of vehicle on a banked road.

Discussion

(a) The velocity (max. possible) of a car on a flat or unbanked road is

$$v_{ub} = \sqrt{\mu_s r g}$$

& on a banked road is $v_b = \left[\dfrac{rg(\mu_s + \tan\theta)}{(1-\mu_s \tan\theta)}\right]^{1/2}$

Clearly $v_b > v_{ub}$

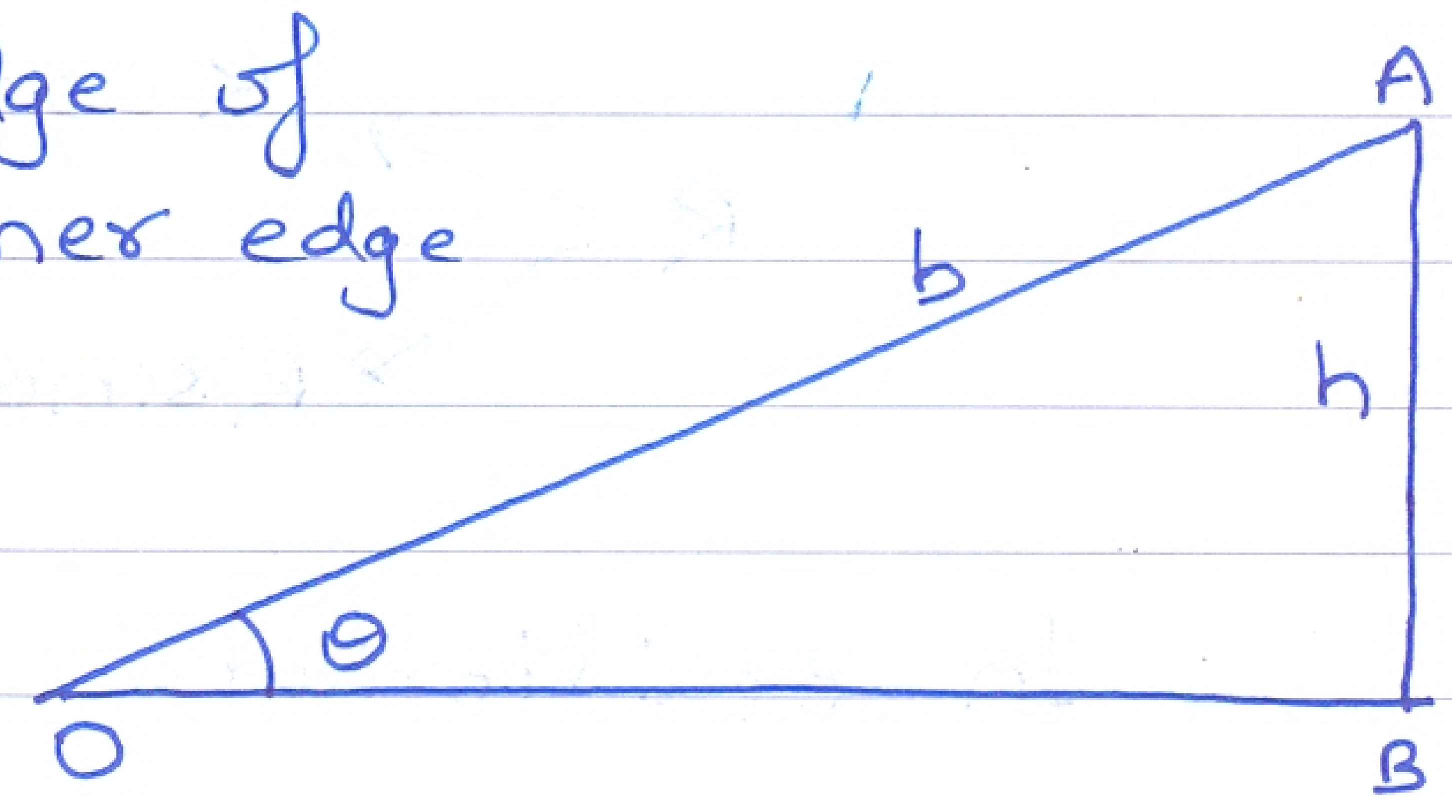
(b) If $\mu_s = 0$ (if banked road is perfectly smooth)

$$v_0 = (rg\tan\theta)^{1/2}$$

At this speed, no frictional force required to provide centripetal force & also no wear & tear of tyres

(c) If speed of vehicle is less than v_0, frictional force will be up the slope.

(d) If h - height of outer edge of road above the inner edge
b - breadth of road



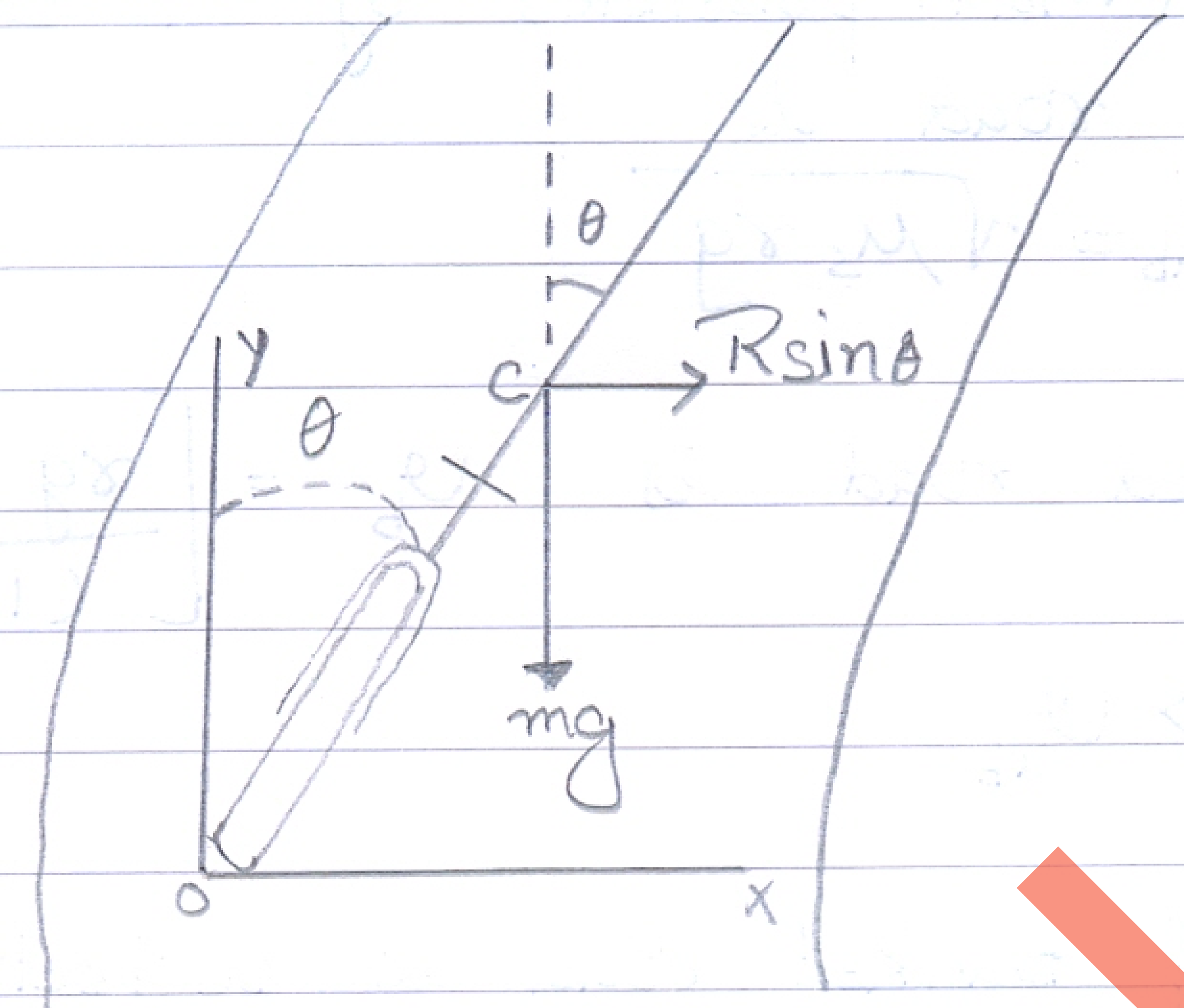
$$\tan\theta = \frac{h}{\sqrt{b^2 - h^2}}$$

or
$$\boxed{\tan\theta = \frac{v_0^2}{rg} = \frac{h}{\sqrt{b^2-h^2}}}$$

Usually $h << b$ so $b^2 - h^2 \approx b^2$

$$\therefore \boxed{\tan\theta = \frac{v_0^2}{rg} = \frac{h}{b}}$$

Bending of a cyclist



Consider a cyclist moving on a curved road.

Let m - mass of cyclist
v - velocity " " while turning
r - radius of the circular path
θ - angle of bending with vertical

Now, mg - weight of cyclist - vertically downwards
R - reaction of ground - at angle θ with vertical

R → $R\cos\theta$ (along vertical upward direction)
→ $R\sin\theta$ (along horizontal towards centre of circular track)

In equilibrium $R\cos\theta = mg$ — (1)

$R\sin\theta = \frac{mv^2}{r}$ — (2) [Rsinθ provides centripetal force]

(2) ÷ (1)
$$\frac{R\sin\theta}{R\cos\theta} = \frac{mv^2}{rmg}$$

$$\boxed{\tan\theta = \frac{v^2}{rg}}$$

For a safe turn, θ should be small