Torque or Moment of force

Torque

- It gives the turning effect of the force about the fixed axis.
- It is measured by the product of magnitude of force and $\perp^r$ distance of the line of action of force from the axis of rotation.

Torque = force × $\perp^r$ distance

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

$$\boxed{\tau = rF\sin\theta\,\hat{n}}$$

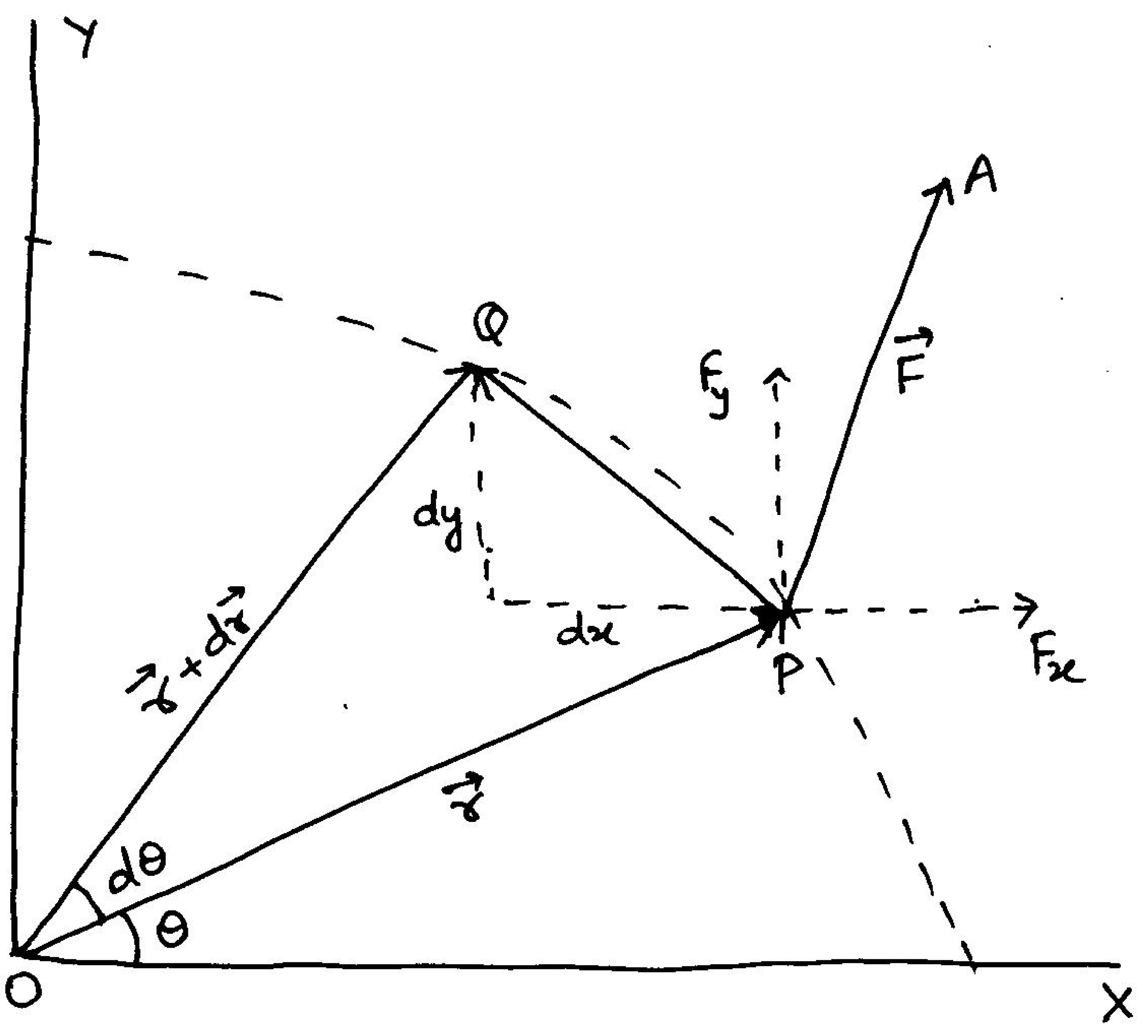
where θ - angle (small) bet^n $\vec{r}$ & $\vec{F}$

$\hat{n}$ - unit vector along $\vec{\tau}$.

S.I. unit - N-m

Dimensions - $[ML^2T^{-2}]$

Expression for torque in cartesian co-ordinates (Physical meaning of torque)



Consider a particle of mass 'm' rotating in plane XY about fixed axis OZ.

Let P be position of the particle at any instant.

$$\vec{OP} = \vec{r}$$

$$\angle XOP = \theta$$

Let rotation occur under the action of force $\vec{F}$ applied at P along $\vec{PA}$

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In a small time dt, let the particle at P reach Q

$$\vec{OQ} = \vec{r} + d\vec{r}$$

$$\angle POQ = d\theta$$

$$\vec{PQ} = d\vec{r}$$

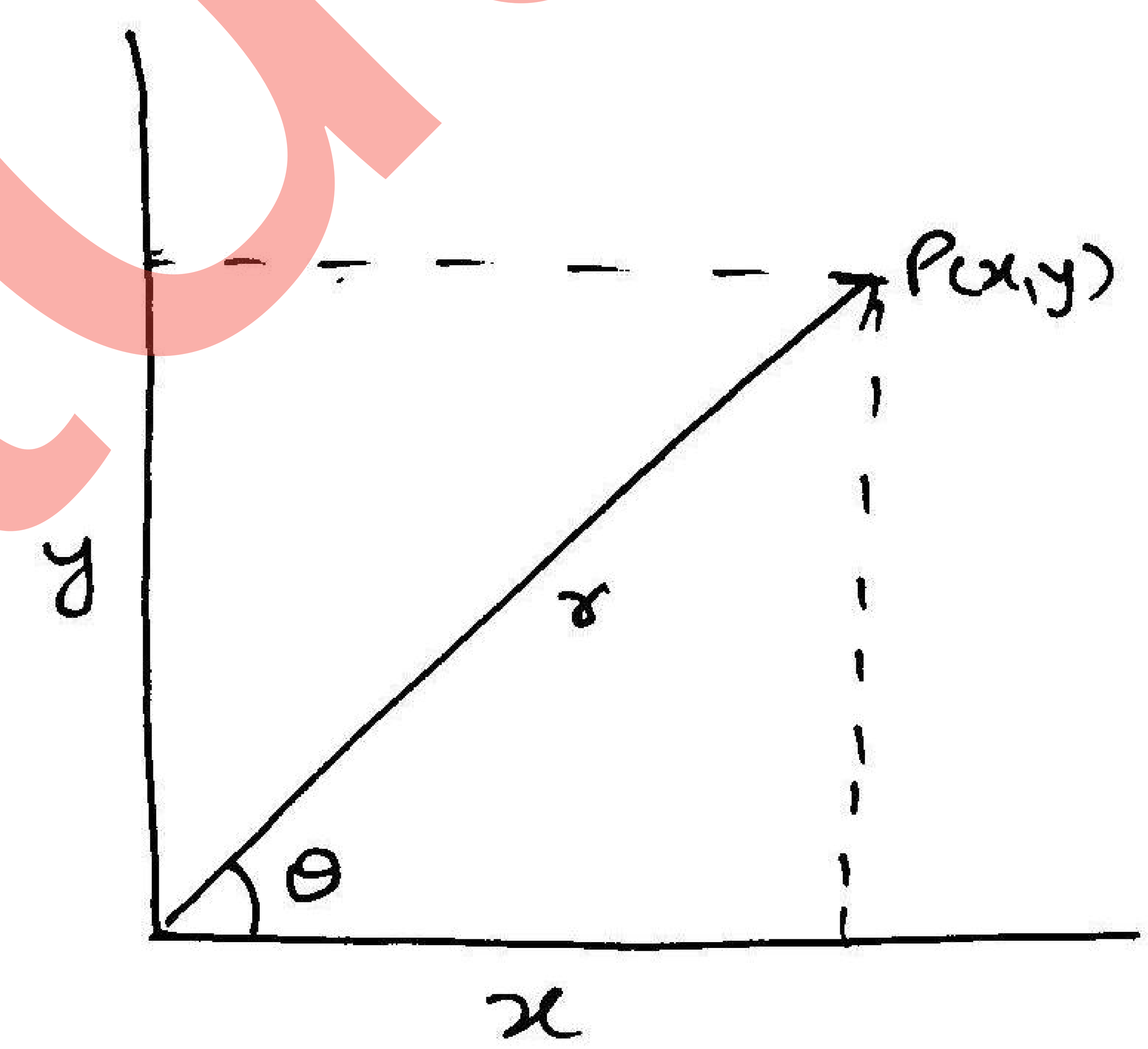
Small work done in rotating the particle from P to Q is

$$dW = \vec{F} \cdot d\vec{r}$$

$$= (\hat{i} F_x + \hat{j} F_y) \cdot (\hat{i}\, dx + \hat{j}\, dy)$$

$$= F_x\, dx + F_y\, dy \quad \text{—— (1)}$$

If (x, y) are the co-ordinates of P, then



$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\frac{dx}{d\theta} = r\frac{d\cos\theta}{d\theta} = -r\sin\theta = -y$$

$$dx = -y\, d\theta$$

$$\frac{dy}{d\theta} = r\frac{d\sin\theta}{d\theta} = r\cos\theta = x$$

$$dy = x\, d\theta$$

$\therefore$ eqn (1) becomes

$$dW = F_x(-y\,d\theta) + F_y(x\,d\theta)$$

$$dW = (xF_y - yF_x)\, d\theta$$

$$dW = \tau \cdot d\theta$$

where $\boxed{\tau = xF_y - yF_x}$

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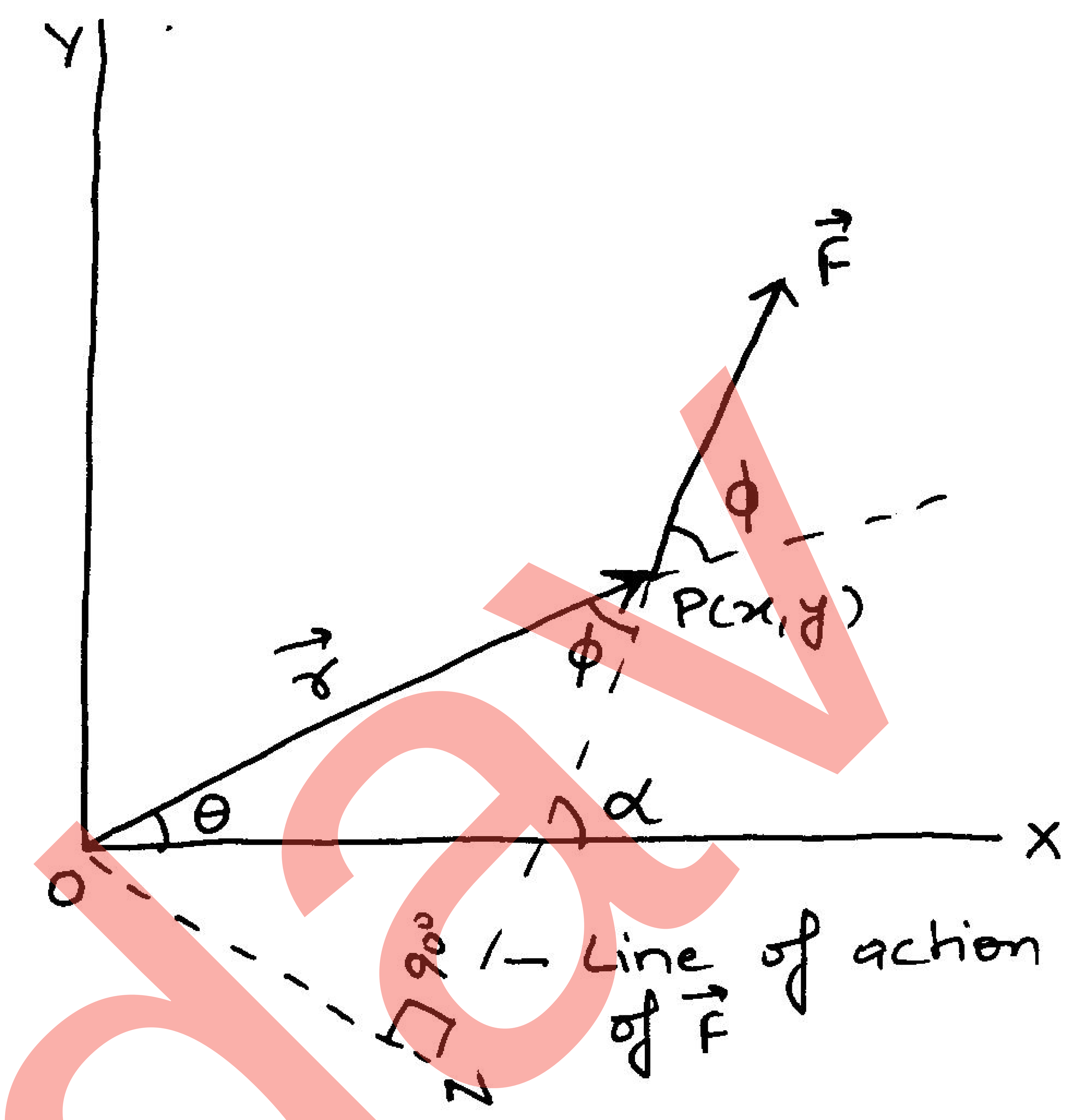
Expression for torque in polar co-ordinates

Let the line of action of $\vec{F}$ makes an angle α with x-axis.



$$\therefore \quad F_x = F\cos\alpha$$
$$F_y = F\sin\alpha$$

If x, y are co-ordinates of P, then

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Now, $\tau = xF_y - yF_x$

$$= (r\cos\theta)F\sin\alpha - r\sin\theta . F\cos\alpha$$
$$= rF[\sin\alpha\cos\theta - \cos\alpha\sin\theta]$$
$$= rF\sin(\alpha - \theta)$$

Let ϕ be the angle which the line of action of $\vec{F}$ makes with OP, then

$$\theta + \phi = \alpha$$
$$\alpha - \theta = \phi$$

$$\therefore \quad \boxed{\tau = rF\sin\phi} \Rightarrow \boxed{\tau = \vec{r} \times \vec{F}}$$

In ΔOPN, $\sin\phi = \frac{ON}{r} \Rightarrow ON = r\sin\phi$

$$\therefore \quad \boxed{\tau = F(ON) = \text{Force} \times \perp^r \text{ distance of line of action of } \vec{F} \text{ from the axis of rotation}}$$

Discussion

① τ is max. when r is max.

eg: (i) Handle is provided near the free edge of the plank of the door.

(ii) To unscrew a nut fitted tightly to a bolt, we need a wrench with long arm.

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② When r is long, F required to produce a given turning effect will be small.

③ $\phi = 90^\circ$, $\sin\phi = +1$ (max) $\Rightarrow$ τ - max. (eg. of door)
$= 0^\circ$ or 180°, $\sin\phi = 0$ $\Rightarrow$ τ - min.

Rectangular components of torque

$$\tau_x = yF_z - zF_y$$
$$\tau_y = zF_x - xF_z$$
$$\tau_z = xF_y - yF_x$$

Power associated with torque

Small work done in rotating a particle through a small angle ($d\theta$) is

$$dW = \tau\, d\theta$$

$$\frac{dW}{dt} = \tau \frac{d\theta}{dt}$$

$$\boxed{P = \tau\omega}$$

where P - average power associated with torque
ω - " angular speed of the body in this interval.

Expression for angular momentum in cartesian co-ordinates

Torque rotating a particle in XY plane is

$$\tau = xF_y - yF_x \quad \text{—— (1)}$$

Acc. to Newton's 2nd law

$$F_x = \frac{dp_x}{dt} = \frac{d(mv_x)}{dt} = m\frac{dv_x}{dt}$$

$$F_y = m\frac{dv_y}{dt}$$

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$$\therefore \quad \tau = xm\frac{dv_y}{dt} - ym\frac{dv_x}{dt}$$

$$= m\left[x\frac{dv_y}{dt} - y\frac{dv_x}{dt}\right] \quad \text{—— (2)}$$

Now, $\frac{d}{dt}(xv_y - yv_x) = \frac{d}{dt}(xv_y) - \frac{d}{dt}(yv_x)$

$$= x\frac{dv_y}{dt} + v_y\frac{dx}{dt} - y\frac{dv_x}{dt} - v_x\frac{dy}{dt}$$

$$= x\frac{dv_y}{dt} + \cancel{v_y v_x} - y\frac{dv_x}{dt} - \cancel{v_x v_y}$$

$$= x\frac{dv_y}{dt} - y\frac{dv_x}{dt}$$

So eqn. (2) becomes

$$\tau = m\frac{d}{dt}(xv_y - yv_x)$$

$$= \frac{d}{dt}(xmv_y - ymv_x)$$

$$= \frac{d}{dt}(xp_y - yp_x)$$

$$\therefore \quad \boxed{\tau = \frac{dL}{dt}}$$

where $L = xp_y - yp_x$
$\rightarrow$ angular momentum

So, Angular momentum of a body about a given axis is the moment of linear momentum of the body about that axis.

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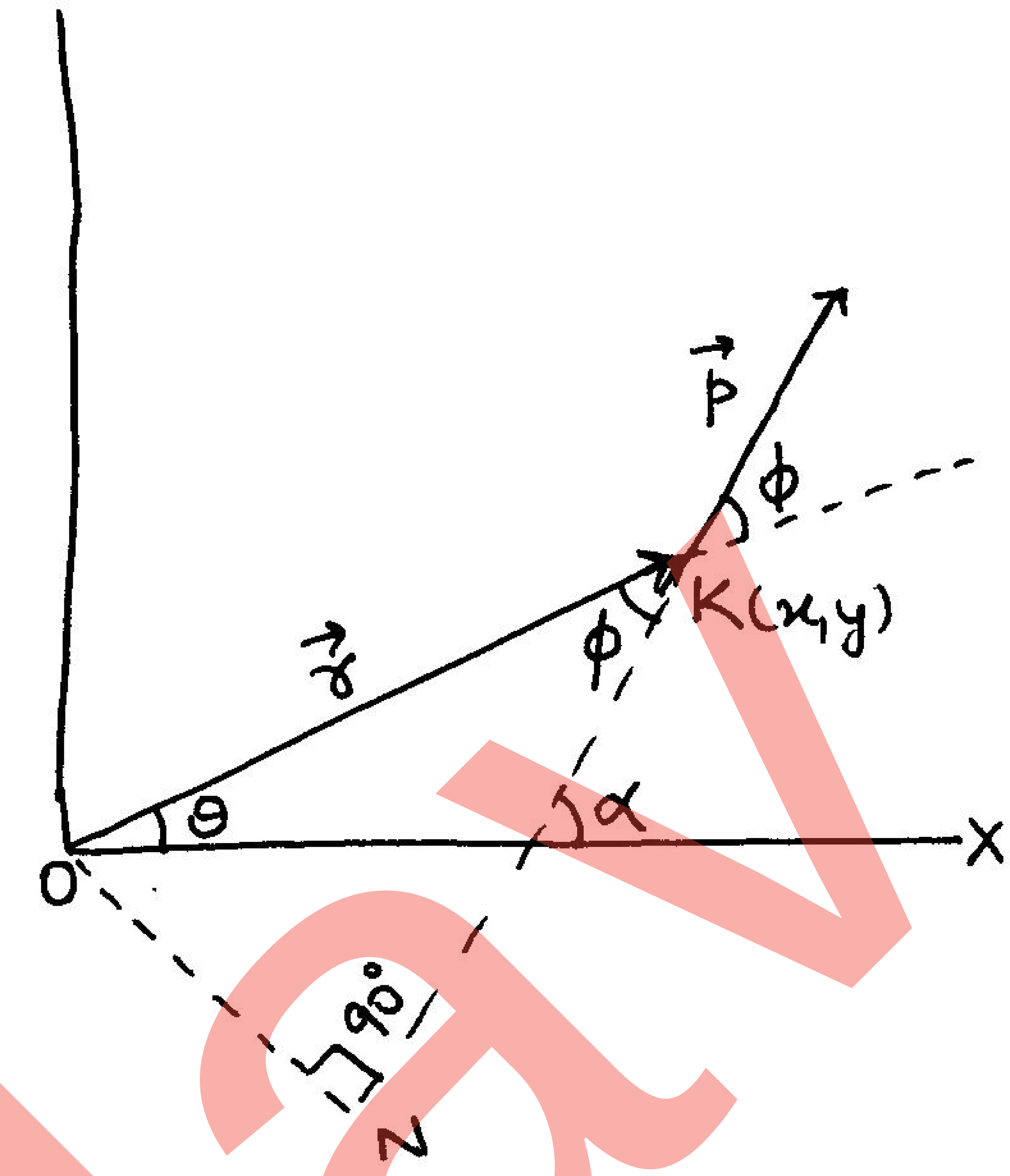
Expression for angular momentum in polar co-ordinates



Suppose $K(x,y)$ be the position of a particle of mass 'm' & linear momentum $\vec{p}$ rotating in XY plane about Z-axis.

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Let the line of action of $\vec{p}$ makes an angle α with OX & angle ϕ with $\vec{r}$.

$\therefore\ p_x = p\cdot\cos\alpha \quad \& \quad p_y = p\sin\alpha$

Now, $L = xp_y - yp_x$

$$= r\cos\theta . p\sin\alpha - r\sin\theta\, p\cos\alpha$$
$$= rp\,[\sin\alpha\cos\theta - \cos\alpha\sin\theta]$$
$$= rp\,\sin(\alpha-\theta)$$
$$\boxed{L = rp\sin\phi} \boxed{= \vec{r}\times\vec{p}} \qquad [\because \phi = \alpha-\theta]$$

In ΔOKN, $\sin\phi = \dfrac{ON}{r} \Rightarrow ON = r\sin\phi$

$\therefore$ $\boxed{L = p\,(ON) = \text{linear momentum} \times \perp^r \text{ distance of line of action of } \vec{p} \text{ vector from axis of rotation}}$

physical meaning of angular momentum

* S.I. unit of L → $kg\,m^2 s^{-1}$
* D.F " " $[ML^2T^{-1}]$
* Vector quantity, direction is given by right hand screw rule.
 " " thumb " .

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Geometrical meaning of angular momentum

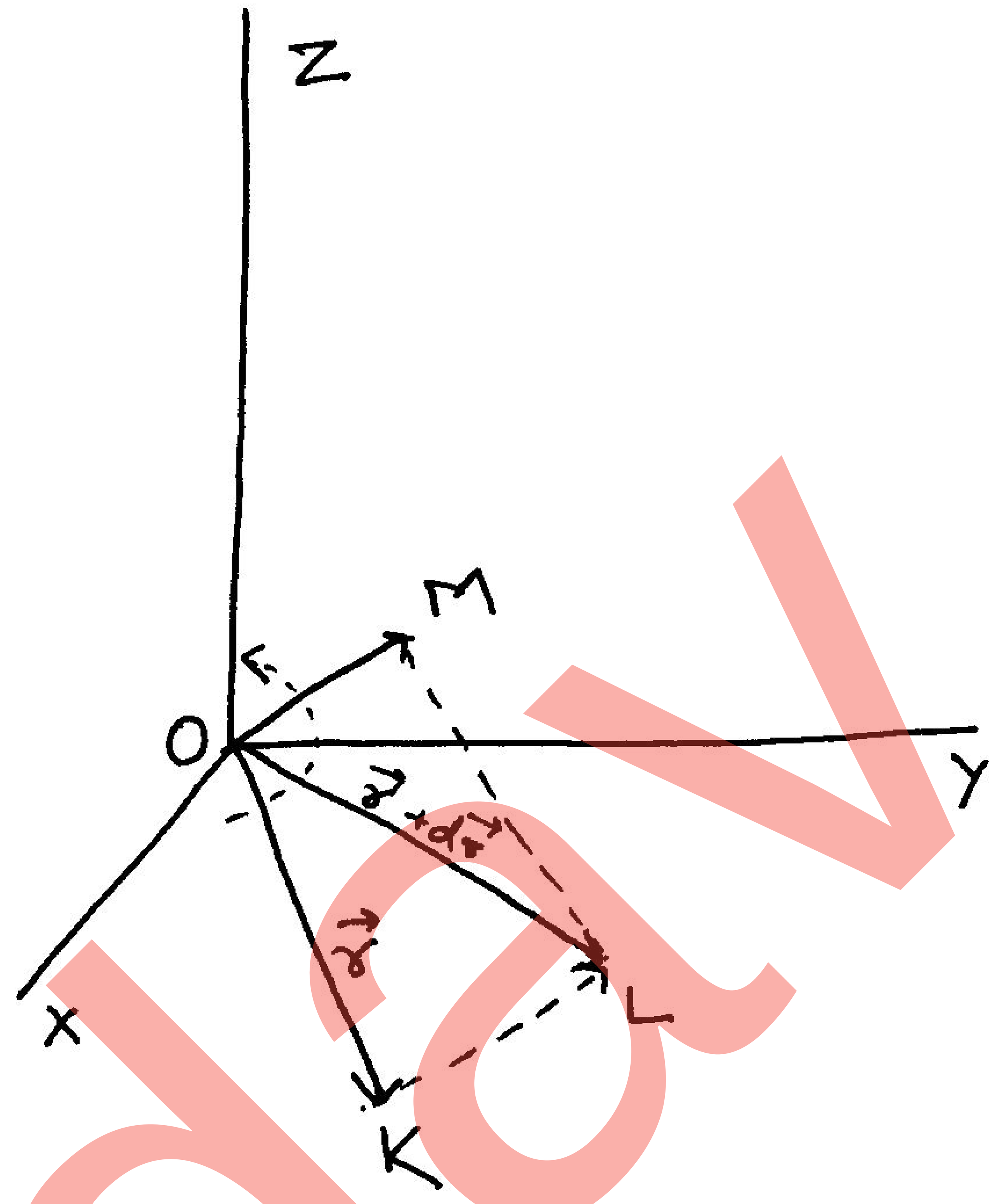
Consider a particle rotating in XY plane about axis OZ.

At any time t, let $\vec{OK} = \vec{r}$

In small time dt, let the particle at K reach L so

$$\vec{OL} = (\vec{r} + d\vec{r})$$

& $$\vec{KL} = d\vec{r}$$



Area swept by the position vector in a small time 'dt' is

$$|d\vec{A}| = \text{Area of } \Delta OKL$$

$$= \frac{1}{2}(\text{Area of } OKLM)$$

$$= \frac{1}{2}(\vec{OK} \times \vec{OM})$$

$$|d\vec{A}| = \frac{1}{2}|\vec{r} \times d\vec{r}|$$

$$\left|\frac{d\vec{A}}{dt}\right| = \frac{1}{2}\left|\vec{r} \times \frac{d\vec{r}}{dt}\right|$$

$$= \frac{1}{2}\left|\vec{r} \times \frac{\vec{p}}{m}\right| \qquad \left[\because \frac{d\vec{r}}{dt} = \vec{v} = \frac{\vec{p}}{m}\right]$$

$$\left|\frac{d\vec{A}}{dt}\right| = \frac{1}{2m}|\vec{L}|$$

or $$\boxed{|\vec{L}| = 2m\left|\frac{d\vec{A}}{dt}\right|}$$

where $\left|\frac{d\vec{A}}{dt}\right|$ – (area swept by the position vector per unit time) Areal velocity of position vector of particle.

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So, angular momentum of a particle about a given axis is twice the product of mass of the particle & areal velocity of position vector of the particle. This is the geometrical meaning of angular momentum.

Relation betⁿ $\vec{L}$ & $\vec{\tau}$

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \vec{v} \times m\vec{v} + \vec{r} \times \vec{F}$$

$$= 0 + \vec{\tau}$$

$$\boxed{\frac{d\vec{L}}{dt} = \vec{\tau}}$$

Equilibrium of rigid bodies

For mechanical equilibrium of a rigid body, 2 conditions need to be satisfied.

① First condition of equilibrium

A rigid body is said to be in translational equilibrium, if it remains at rest or moving with a constant velocity in a particular direction.

For this, the net external force or the vector sum of all the external forces acting on the body must be zero.

i.e. $\vec{F} = 0$

$$m\vec{a} = 0$$

$$a = 0$$

$$\Rightarrow \quad v = \text{constant}$$

$$\Rightarrow \quad p - \text{constant.}$$

* body at rest - static equilibrium
  in uniform motion - dynamic 〃

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Also, $F = -\frac{dU}{dr}$, where U - potential energy of the body

In translational equilibrium, $F = 0$

$$-\frac{dU}{dr} = 0$$

$$\boxed{U = \text{constant}}$$

Translational static equilibrium

Stable equilibrium

1. Body tries to regain its equilibrium position after being slightly displaced & displaced.
2. U tends to be min.
3. chair lying on ground.

Unstable equilibrium

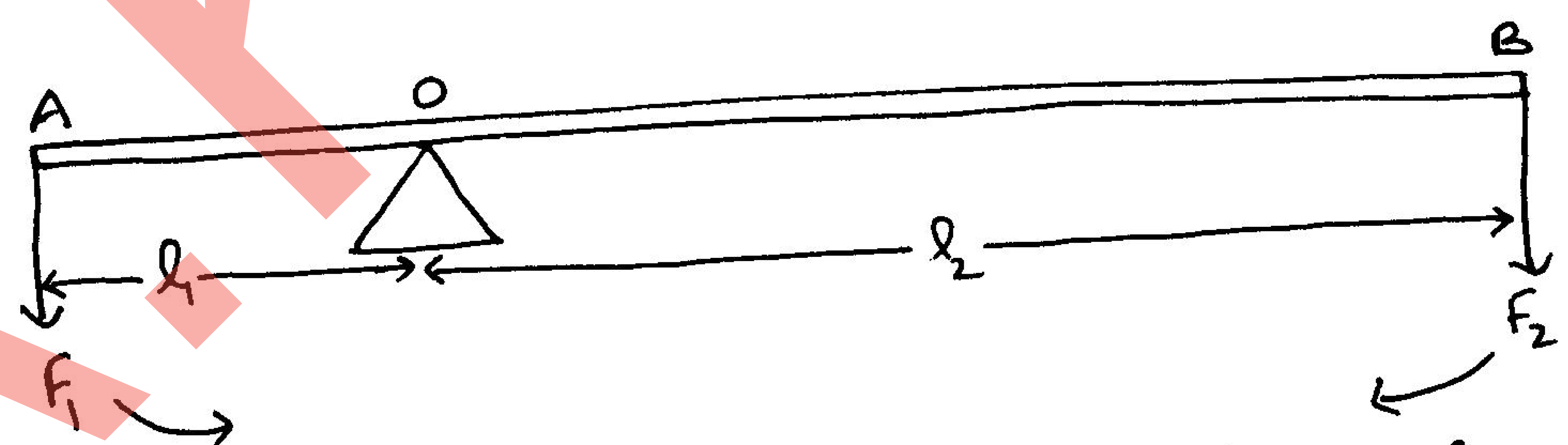
1. Body gets disturbed further after being slightly displaced and released
2. U tends to increase
3. book standing on edge

Neutral equilibrium

1. Body can stay in equilibrium even after being slightly displaced and released.
2. No change in U
3. ball rolling on ground.

(2) Second condition of equilibrium

A rigid body is said to be in rotational equilibrium, if the body does not rotates with const. angular velocity. For this, the net external torque acting on the body is 0.



This beam-balance will be in rotational equilibrium, if

$$F_1 \times l_1 - F_2 \times l_2 = 0 \qquad [\because \text{anticlockwise moment} = +ve]$$

$$\tau_1 + \tau_2 = 0$$

$$\boxed{\Sigma \tau = 0} \Rightarrow \boxed{L = \text{constant}}$$

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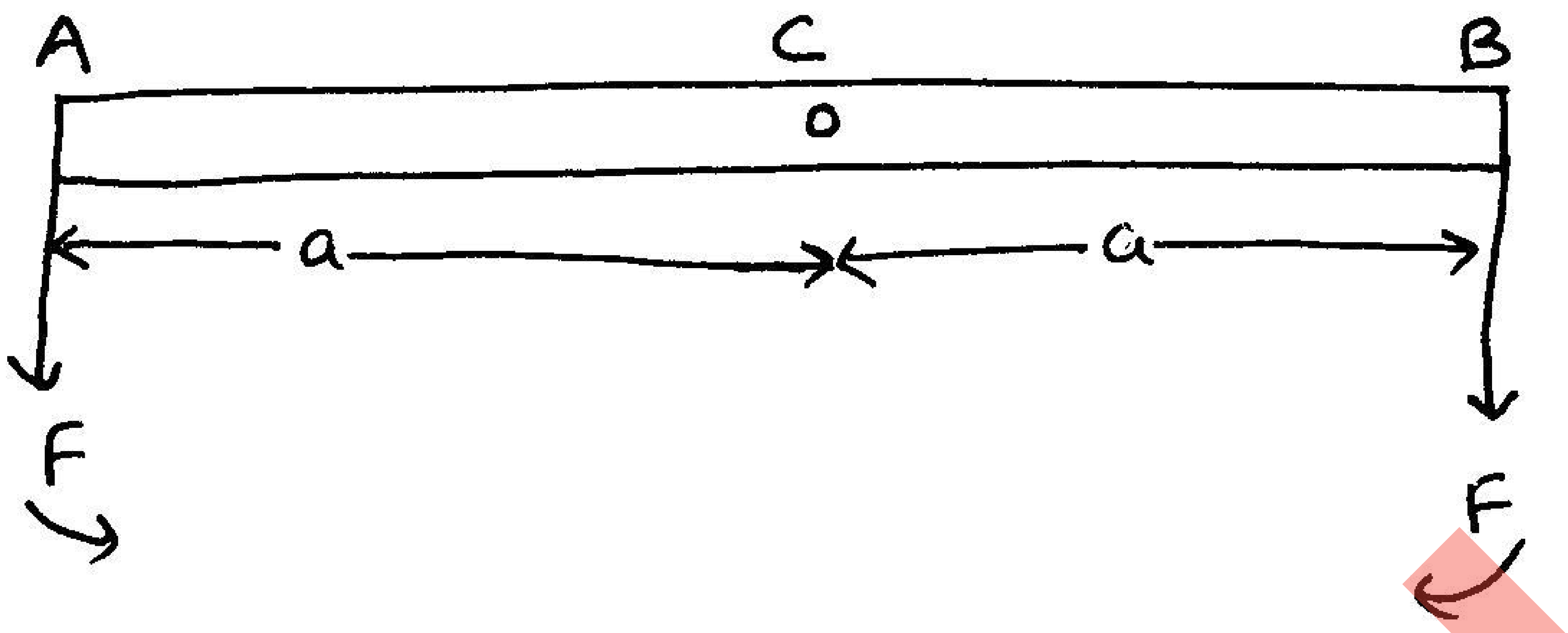
* Partial equilibrium

The rigid body may be in translational equilibrium & not in rotational equilibrium. or vice-versa.

eg (1) rotational but not translational



Net external force $= F + F = 2F \neq 0$

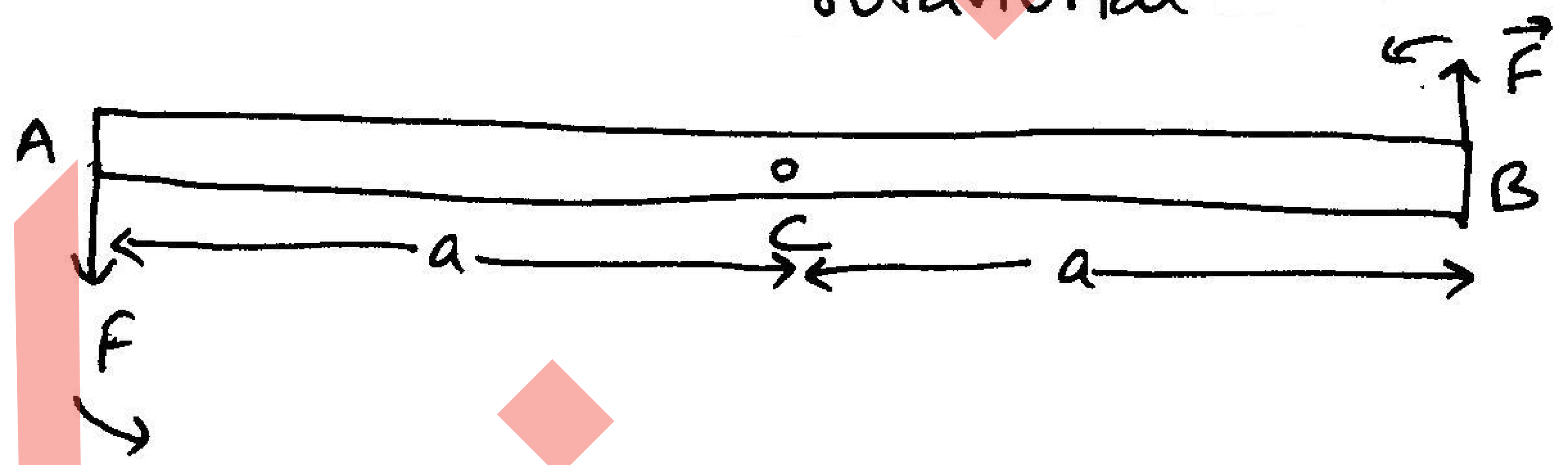
As $\sum \vec{F} \neq 0$ [no translational equilibrium]

Moment of force at A about fixed pt. C $= aF$

" " " " B " " " " $= -aF$

$\therefore \quad \sum \tau = 0$ [rotational equilibrium]

(2) translational but not rotational



Net external force $= F - F = 0$ [translational equilibrium]

Net moment $= aF + aF = 2aF \neq 0$

As $\sum \tau \neq 0$ [no rotational equilibrium]

Couple – A pair of equal & opposite forces with different lines of action are said to form a couple.

eg (1) When we open the lid of a bottle by turning it, our fingers apply a couple on the lid.

(2) When a compass needle is held arbitrarily, it aligns itself in N-S direction because of the couple acting on it.

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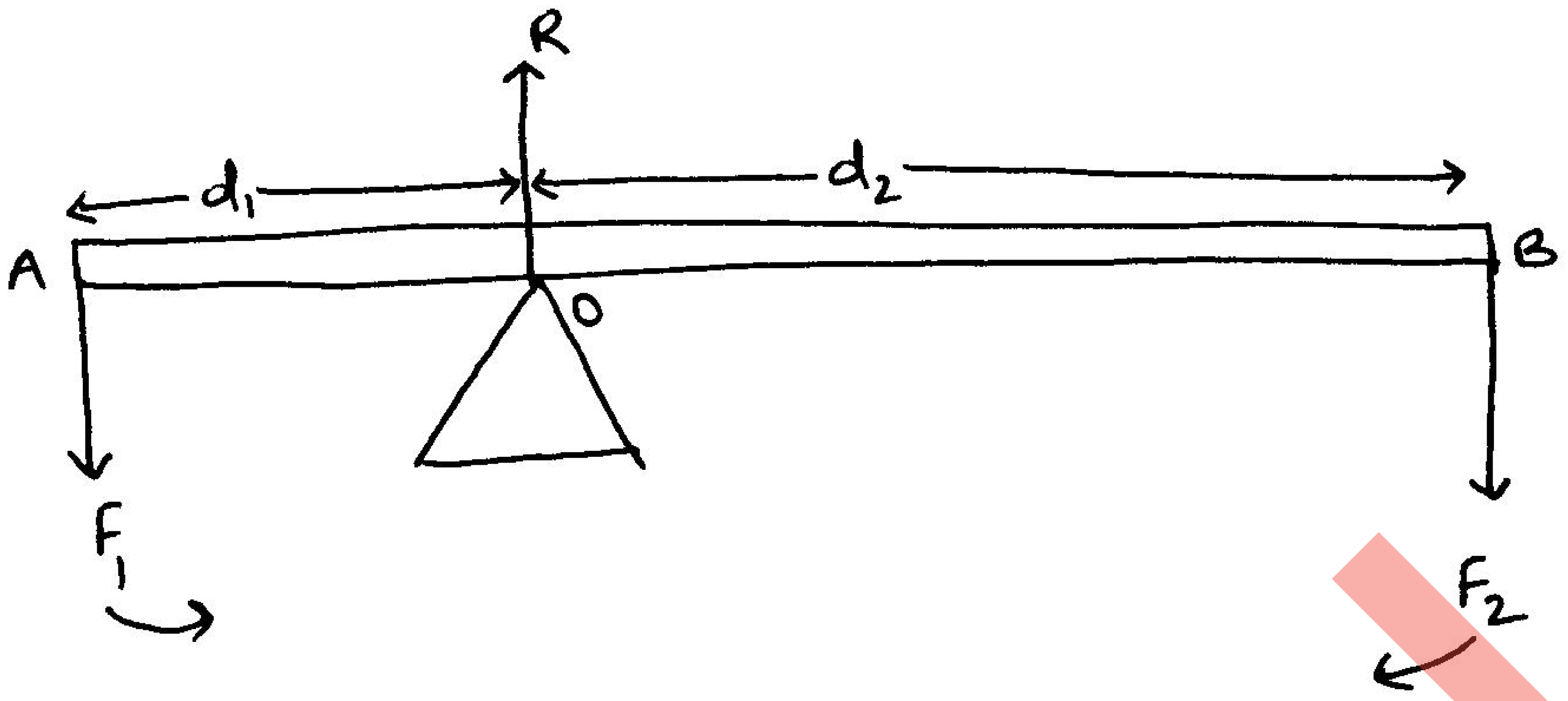
Principle of moments

A body will be in rotational equilibrium if algebraic sum of the moments of all forces acting on the body, about a fixed pt. is zero.



Consider an ideal lever comprising of light rod AB of negligible mass pivoted at pt. O (fulcrum)

Let F_1 – force applied at A

F_2 – " " " B

R – reaction of the support at fulcrum.

As lever is a system in mechanical equilibrium, so it should have translational & rotational equilibrium.

For translational equilibrium

$$\text{net force} = 0$$

$$\therefore \quad R - F_1 - F_2 = 0$$

$$R = F_1 + F_2$$

For rotational equilibrium

$$F_1 \times d_1 - F_2 \times d_2 = 0$$

$$F_1 \times d_1 = F_2 \times d_2$$

A.K. Yadav