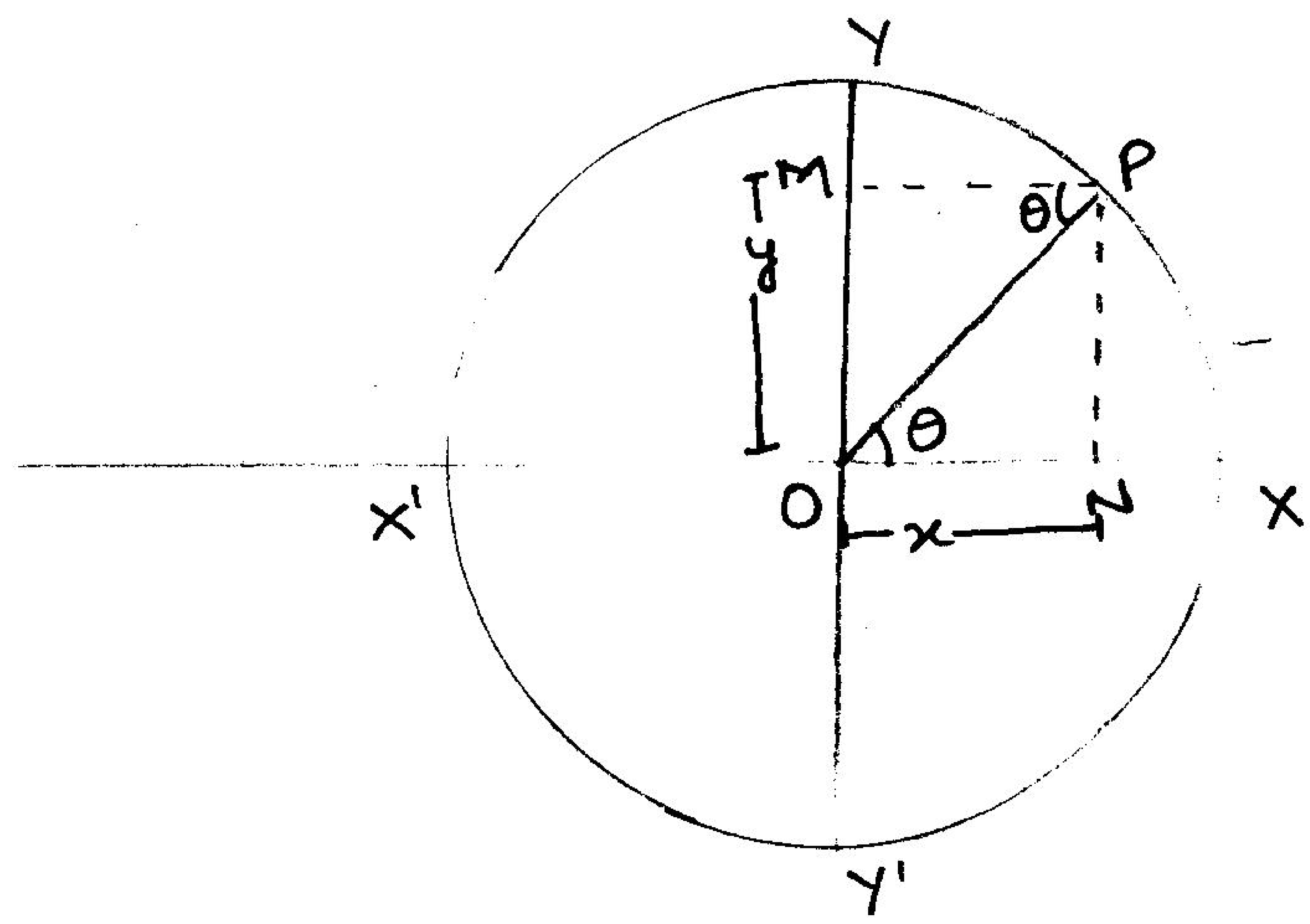
Geometrical Interpretation of S.H.M.



P – reference particle
O – centre of circle of reference
X'OX, Y'OY – diameter " " "

- When P moves from X to Y, its projection on YOY' moves from O to Y.
- When P moves from Y to X', its projection moves from Y to O.
- Similarly, from X' to X, P moves from O to Y' & Y' to O.
- So, during the time when P completes one revolution, its projection M moves to & fro about O along YOY' and completes one vibration with O as mean position.
- The motion of M on YOY' is called S.H.M

"S.H.M. is defined as the projection of a uniform circular motion on any diameter of circle of reference."

Characteristics of S.H.M

① Displacement – Distance of the particle from mean position at that instant.

In ΔOPM, $\sin\theta = \frac{y}{a}$, $\boxed{y = a\sin\theta = a\sin\omega t}$

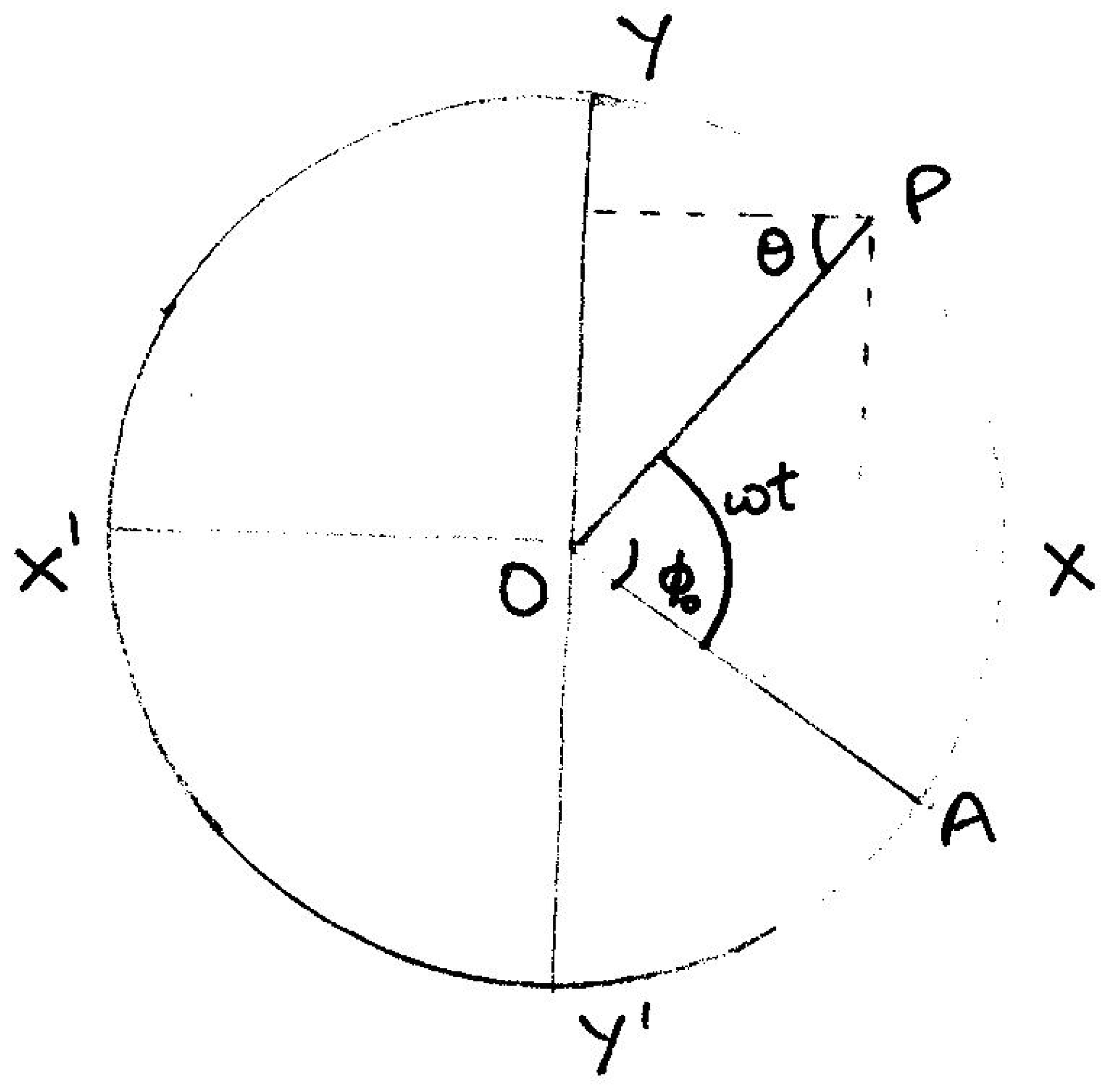
- If P is taken on XOX'

$$\cos\theta = \frac{x}{a}$$

$$\boxed{x = a\cos\theta = a\cos\omega t}$$

- If A is the starting point such that

$\angle AOX = \phi_0$ $\quad \angle AOP = \omega t$



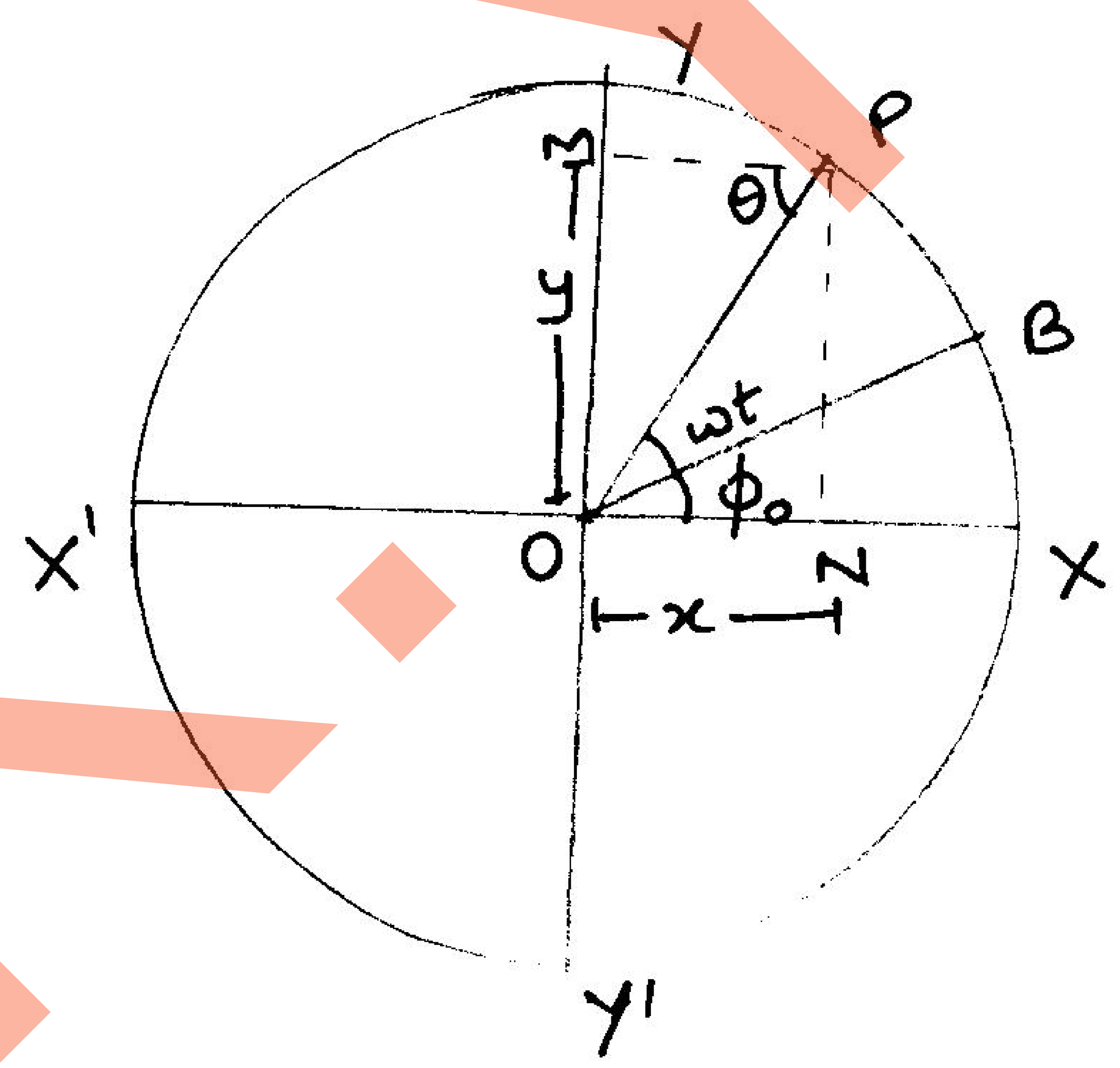
$\theta = \omega t - \phi_0$ [ϕ_0 - initial phase or epoch]

$y = a \sin(\omega t - \phi_0)$

$x = a \cos(\omega t - \phi_0)$

- If B is the starting point such that

$\angle BOX = \phi_0$, $\angle BOP = \omega t$



$\theta = \omega t + \phi_0$

$y = a \sin(\omega t + \phi_0)$

$x = a \cos(\omega t + \phi_0)$

② Amplitude - Max. displacement on either side of mean position.

In S.H.M , max. $\sin\theta$ or $\cos\theta = 1$

$\therefore \; x = y = a$

③ Velocity

$$V = \frac{dy}{dt}$$

$$= \frac{d}{dt}[a \sin(\omega t + \phi_0)]$$

$$= a \cos(\omega t + \phi_0) \times \omega$$

$$= a\omega\sqrt{1 - \sin^2(\omega t + \phi_0)}$$

$$= a\omega\sqrt{1 - y^2/a^2}$$

$$\boxed{V = \omega\sqrt{a^2 - y^2}}$$

At mean position, $y = 0$, $V = a\omega$ (max.)

" extreme ", $y = a$, $V = 0$ (min.)

* direction of velocity – either towards or away from mean position.

④ Acceleration

$$A = \frac{dV}{dt}$$

$$= \frac{d}{dt}[a\omega \cos(\omega t + \phi_0)]$$

$$= a\omega(-\sin(\omega t + \phi_0))\omega$$

$$= -\omega^2 a \sin(\omega t + \phi_0)$$

$$= -\omega^2 y$$

At mean position, $y = 0$, $A = 0$ (min.)

extreme ", $y = a$, $A = -\omega^2 a$ (max.)

* As displacement increases in the direction away from mean position so acceleration is always directed towards mean position.

$$A \propto y$$

"A particle is said to be executing S.H.M if its acceleration at any instant is directly proportional to its displacement from mean position & is always directed towards mean position."

⑤ Time period – Time taken by particle executing SHM to complete one vibration.

$$A = \omega^2 y$$

$$\omega = \sqrt{\frac{A}{y}}$$

Also, $\omega = \frac{2\pi}{T}$

So, $\frac{2\pi}{T} = \sqrt{\frac{A}{y}}$

$$\boxed{T = 2\pi\sqrt{\frac{y}{A}}}$$

⑥ Frequency – No. of vibrations completed in 1 second.

$$\boxed{\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{\text{acceleration}}{\text{displacement}}}}$$

Graphical representation of displacement (y), velocity (v) and acceleration in S.H.M

$$y = a\sin\omega t = a\sin\frac{2\pi}{T}t$$

$$V = a\omega\cos\omega t = a\omega\sin\left(\omega t + \frac{\pi}{2}\right)$$

$$A = -a\omega^2\sin\omega t = a\omega^2\sin(\omega t + \pi)$$

At $t = 0$

$y = 0$

$V = a\omega\cos 0 = a\omega$

$A = 0$

At $t = \frac{T}{4}$

$y = a$

$V = a\omega\sin\pi = 0$

$A = a\omega^2\sin\frac{3\pi}{2}$

$= -a\omega^2$

At $t = \frac{2T}{4}$

$y = 0$

$V = -a\omega$

$A = 0$

At $t = \frac{3T}{4}$

$y = -a$

$V = 0$

$A = a\omega^2$

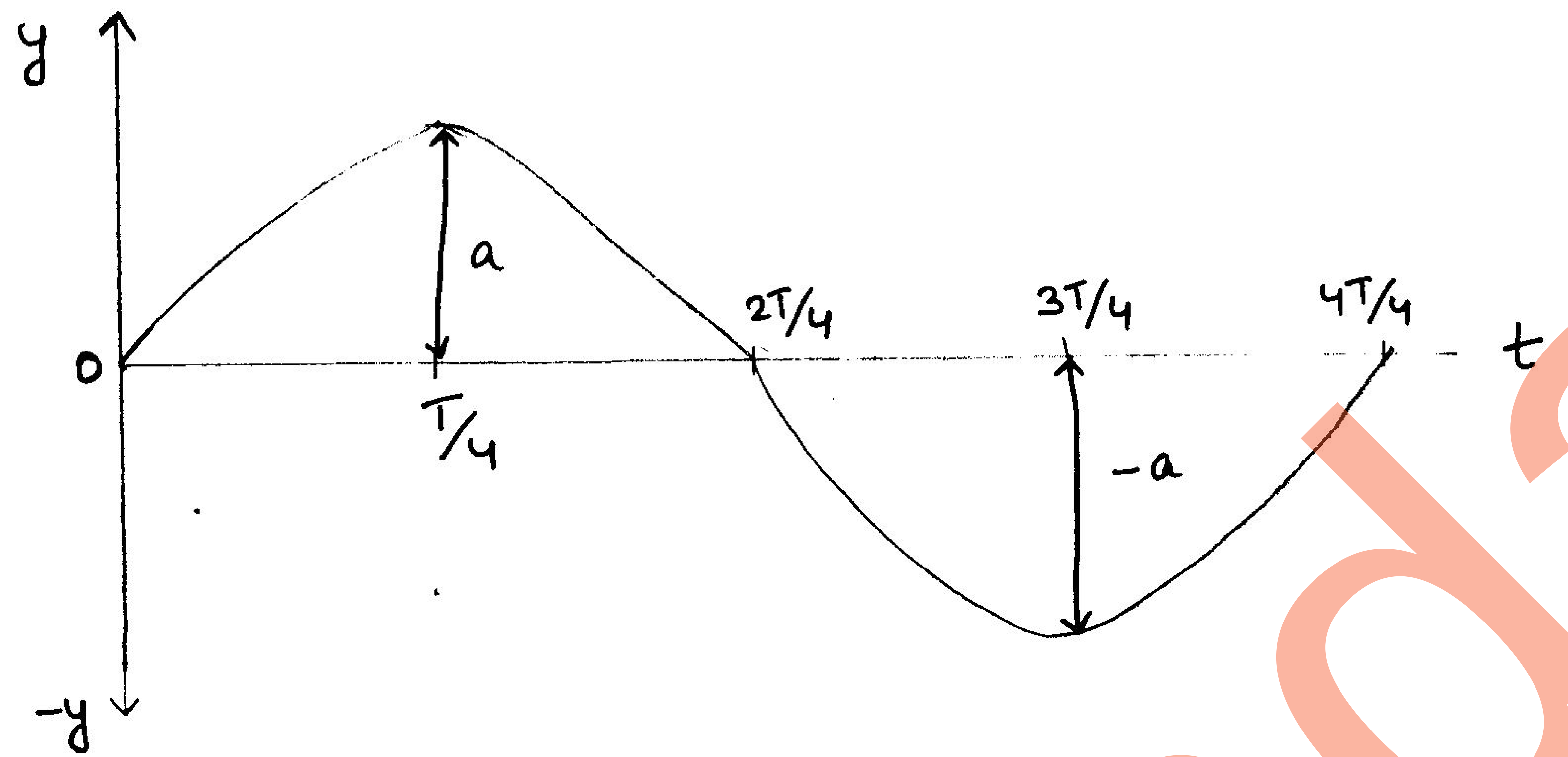
At $t = T$

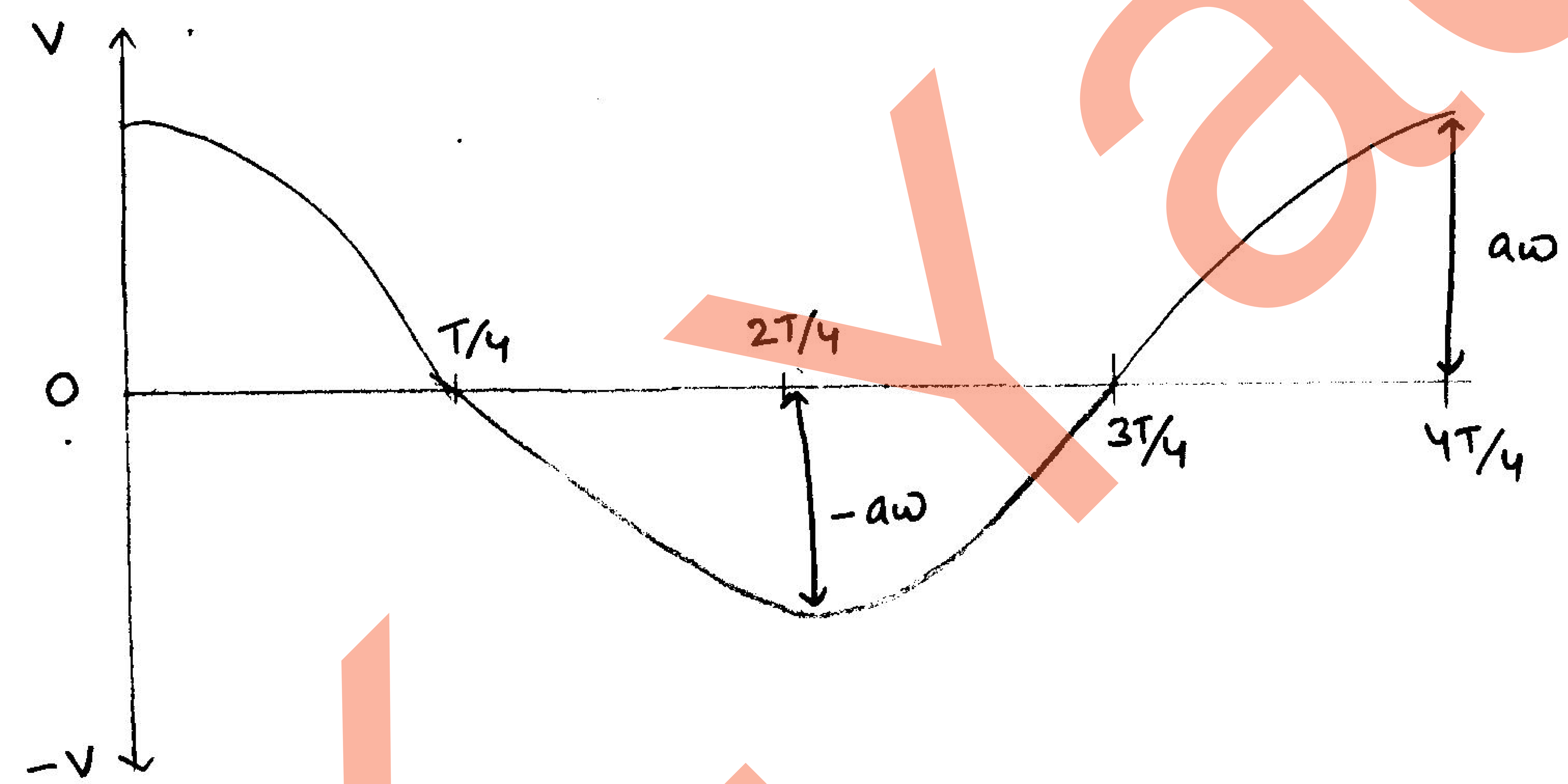
$y = 0$

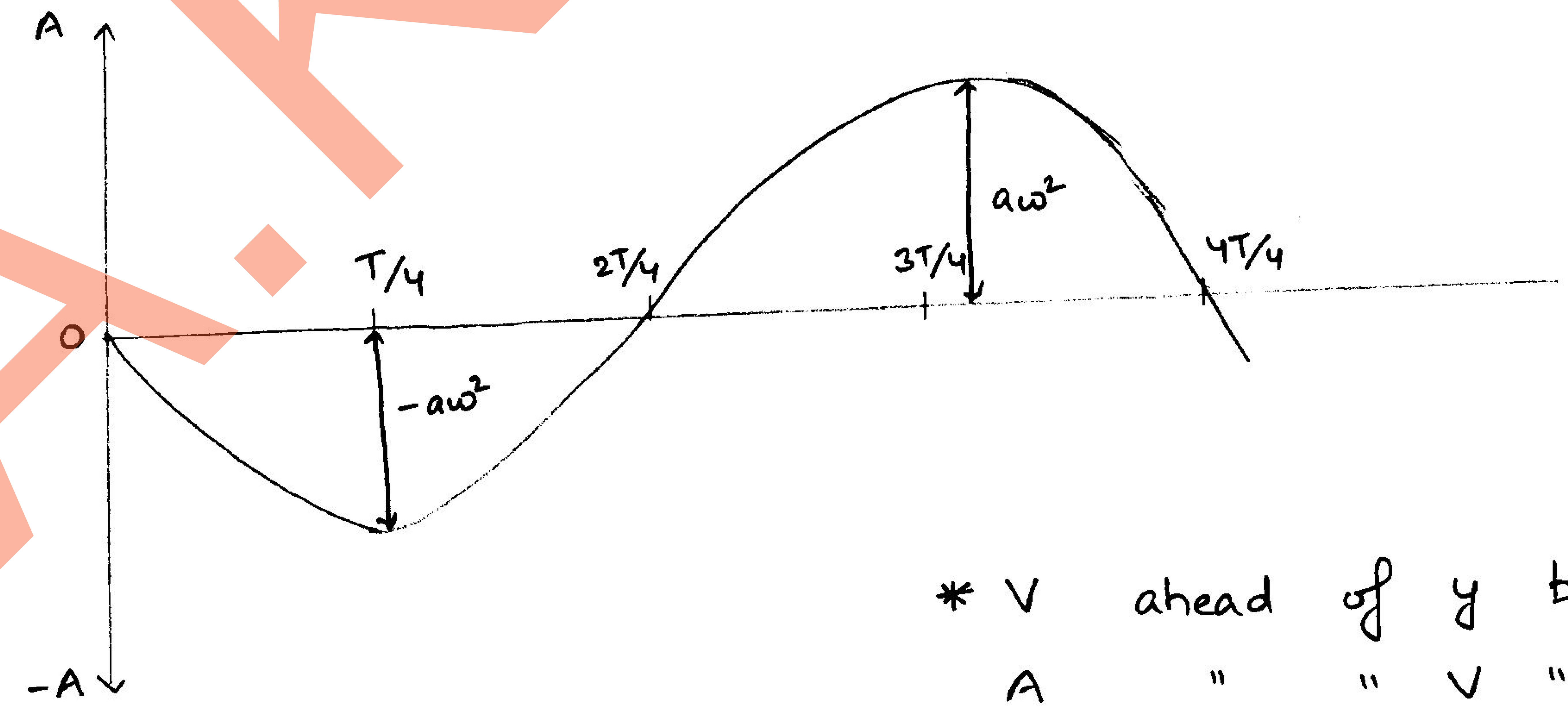
$V = a\omega$

$A = 0$

t	0	T/4	2T/4	3T/4	4T/4
y	0	a	0	−a	0
V	aω	0	−aω	0	aω
A	0	$-a\omega^2$	0	$a\omega^2$	0







* V ahead of y by $\pi/2$

A " " V " $\pi/2$

A " " y " $\pi/2 + \frac{\pi}{2} = \pi$

Total energy in S.H.M

A particle executing S.H.M possess 2 types of energies -

(i) P.E. - due to displacement from its mean position

(ii) K.E - " " velocity of particle

Potential Energy

Consider a particle executing S.H.M having mass 'm', amplitude 'a' & constant angular frequency 'ω'.

The displacement, velocity & acc. of the particle from mean position after time 't' is

$$y = a \sin \omega t$$
$$V = a\omega \cos \omega t$$
$$A = -\omega^2 y$$

The restoring force set-up in the particle is

$$F = mA$$
$$= m(-\omega^2 y)$$
$$= -m\omega^2 y$$
$$F = -ky$$

where $k = m\omega^2$
spring factor of S.H.M.

The work done for very small additional displacement 'dy' against the restoring force is

$$dW = -F\,dy$$
$$= ky\,dy$$

The total work done for displacing the particle from mean position to position of displacement is

$$W = \int_0^y ky\,dy$$

or $$U = \frac{1}{2}ky^2 = \frac{1}{2}m\omega^2 y^2 = \frac{1}{2}m\omega^2 a^2 \sin^2 \omega t$$

Kinetic Energy

$$K = \frac{1}{2}mv^2$$

$$= \frac{1}{2}ma^2\omega^2\cos^2\omega t$$

$$= \frac{1}{2}ka^2\cos^2\omega t \qquad [\because k = m\omega^2]$$

$$= \frac{1}{2}ka^2(1-\sin^2\omega t)$$

$$= \frac{1}{2}ka^2\left(1-\frac{y^2}{a^2}\right)$$

$$= \frac{1}{2}m\omega^2(a^2-y^2)$$

Total Energy

$$E = U + K$$

$$= \frac{1}{2}ky^2 + \frac{1}{2}k(a^2-y^2)$$

$$= \frac{1}{2}ka^2$$

$$\boxed{E = \frac{1}{2}m\omega^2a^2}$$

As for a given particle in S.H.M, m, ω & a are constant so total energy remains constant.

At $t = 0$, $y = 0$

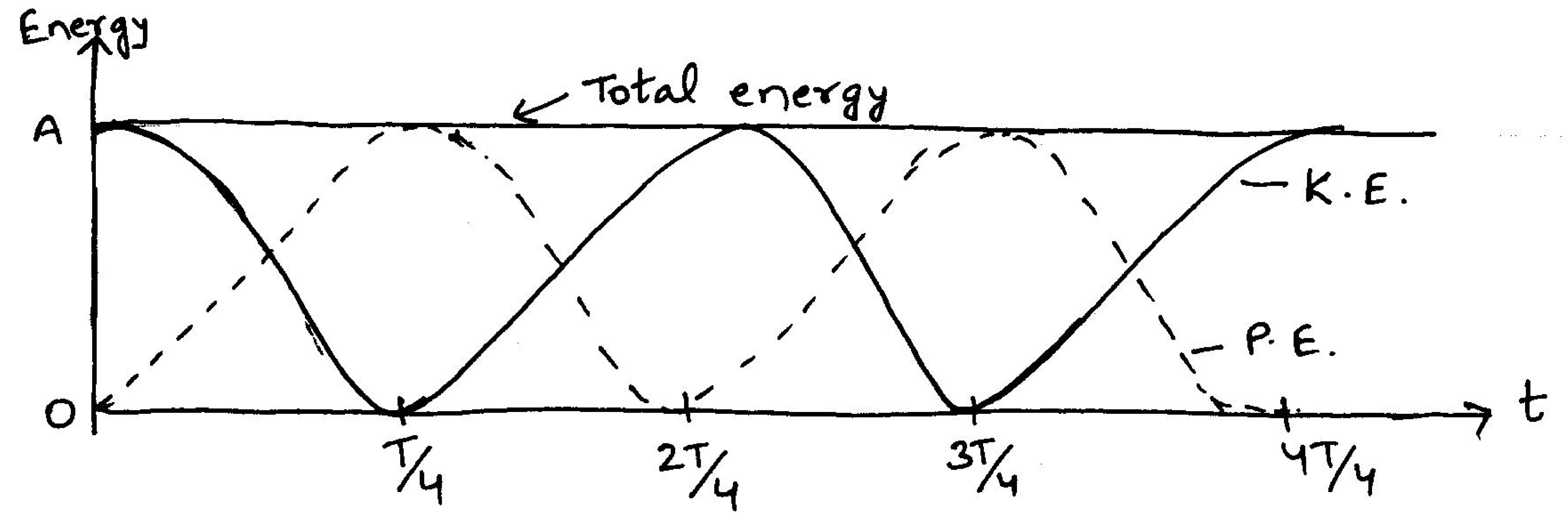
$U = 0$, $K = \frac{1}{2}m\omega^2a^2$, $E = \frac{1}{2}m\omega^2a^2$

i.e. at mean position, total energy is in the form of K.E.

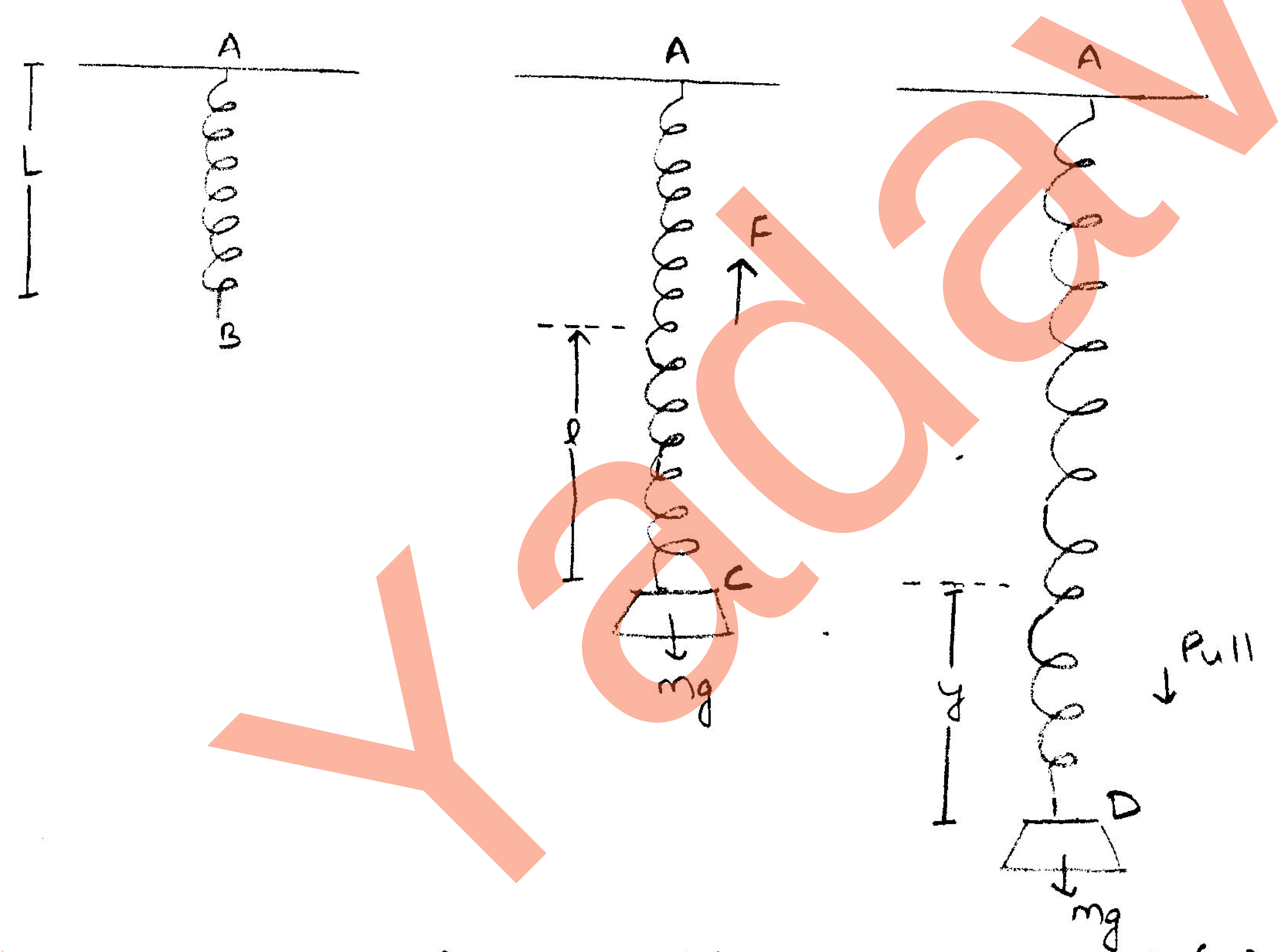
At $t = T/4$, $y = a$

$U = \frac{1}{2}m\omega^2a^2$, $K = 0$, $E = \frac{1}{2}m\omega^2a^2$

i.e. at extreme position, total energy is in the form of P.E.



Motion of a loaded spring



- Consider a spring AB of negligible mass, length 'L' suspended from A.
- When a mass 'm' is attached to the free end, it elongates to C such that $BC = l$ & a restoring force comes into play

$$F = -Kl \quad \text{—— (1)}$$

K – spring factor or force constant

If $l = 1$, $K = F$

"The force constant of a spring may be defined as the restoring force set up per unit elongation in the spring."

At C, the system of the spring is in equilibrium with mass, so

$$F = -mg$$

$$\therefore \; mg = kl$$

Now, displace the attached mass to pt. D through 'y' below C.

Total extension $= l + y$

restoring force $= F'$

So, $F' = -k(l+y)$ —— ②

② − ①

$$F' - F = -k(l+y) - (-kl)$$
$$= -ky$$

If the mass m is released then under the action of $(F' - F)$ the mass will return to C.

If 'a' is the acceleration produced in motion of 'm' so

$$a = \frac{F' - F}{m} = -\frac{k}{m}y$$

As k & m are constants, the acceleration at any instant is proportional to displacement at that time & is always directed towards equilibrium. So, the motion is S.H.M.

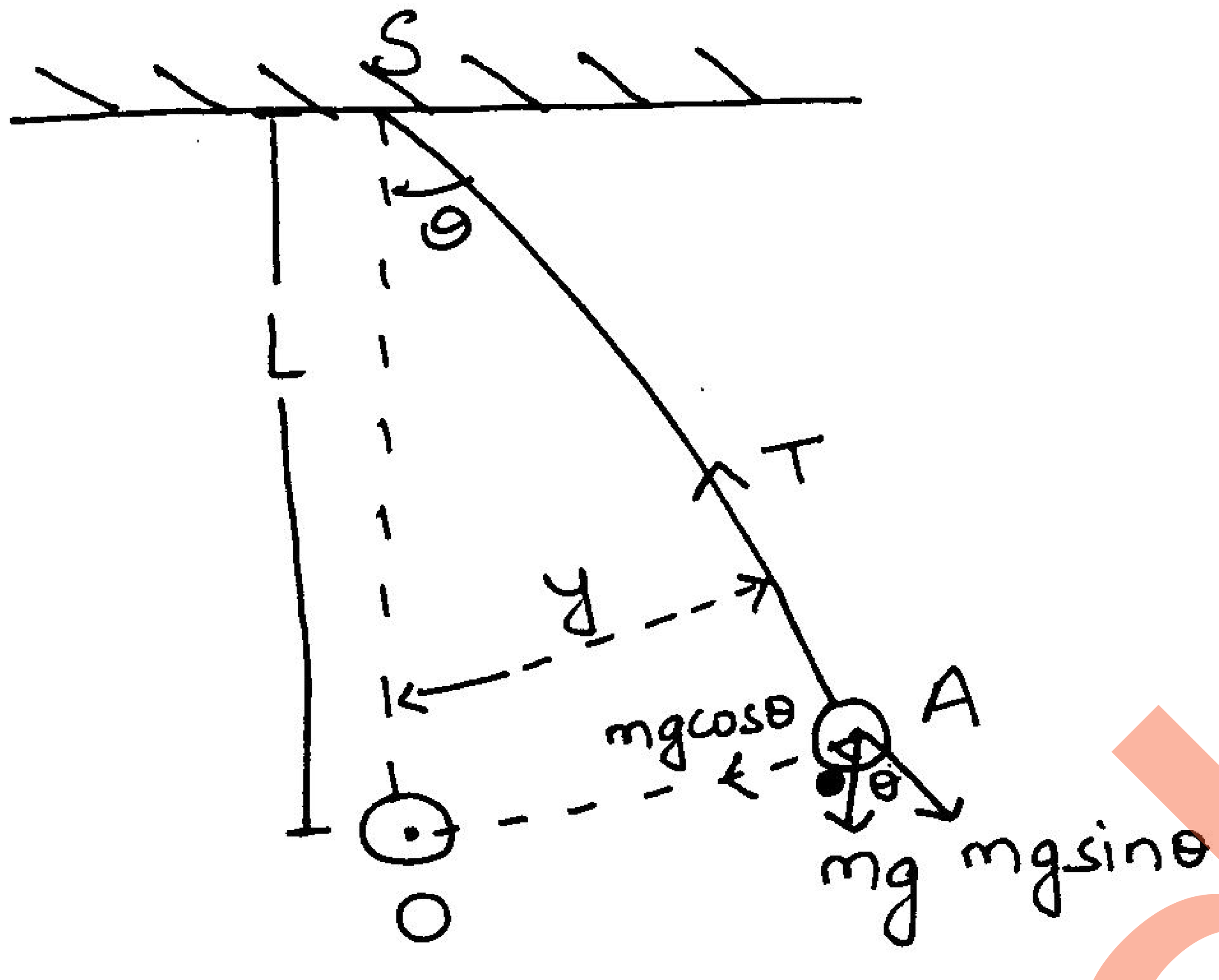
Now, $T = 2\pi\sqrt{\frac{y}{a}}$

$$= 2\pi\sqrt{\frac{m}{k}}$$

$$\boxed{T = 2\pi\sqrt{\frac{l}{g}}}$$

Simple pendulum

It is a heavy point mass suspended by a weightless, inextensible and a perfectly flexible string from a rigid support about which it can vibrate freely.



Suppose a metallic bob of weight 'mg' is suspended from S. with a fine thread OS = L

Let the bob is displaced from O to A such that

$$\widehat{OA} = y \quad \& \quad \theta = \frac{y}{L}$$

When the bob is at A, the forces acting on it are

a) Weight (mg) acting downwards

b) Tension (T) in string acting along its length towards the pt. of suspension.

mg can be resolved into 2 components

(i) $mg\cos\theta$ – balances T

(ii) $mg\sin\theta$ – acts as restoring force on bob

So, $F = -mg\sin\theta$

$$\therefore \quad a = \frac{F}{m} = -g\sin\theta = -g\theta = -g\frac{y}{L} \qquad [\sin\theta \approx \theta]$$

As, g & L are constants, so $a \propto y$ & is directed towards mean position

So, motion is SHM

$$T = 2\pi\sqrt{\frac{y}{a}}$$

$$\boxed{T = 2\pi\sqrt{\frac{L}{g}}}$$

Time period of a pendulum is independent of
(i) mass of bob
(ii) amplitude of vibration, if θ is small

__Second's pendulum__ – pendulum having time period of 2 seconds

$$T = 2\pi\sqrt{\frac{L}{g}}$$

$$L = \frac{T^2 g}{4\pi^2}$$

On putting $T = 2\,\text{sec.}$ & $g = 980\ \text{cm}\,s^{-2}$

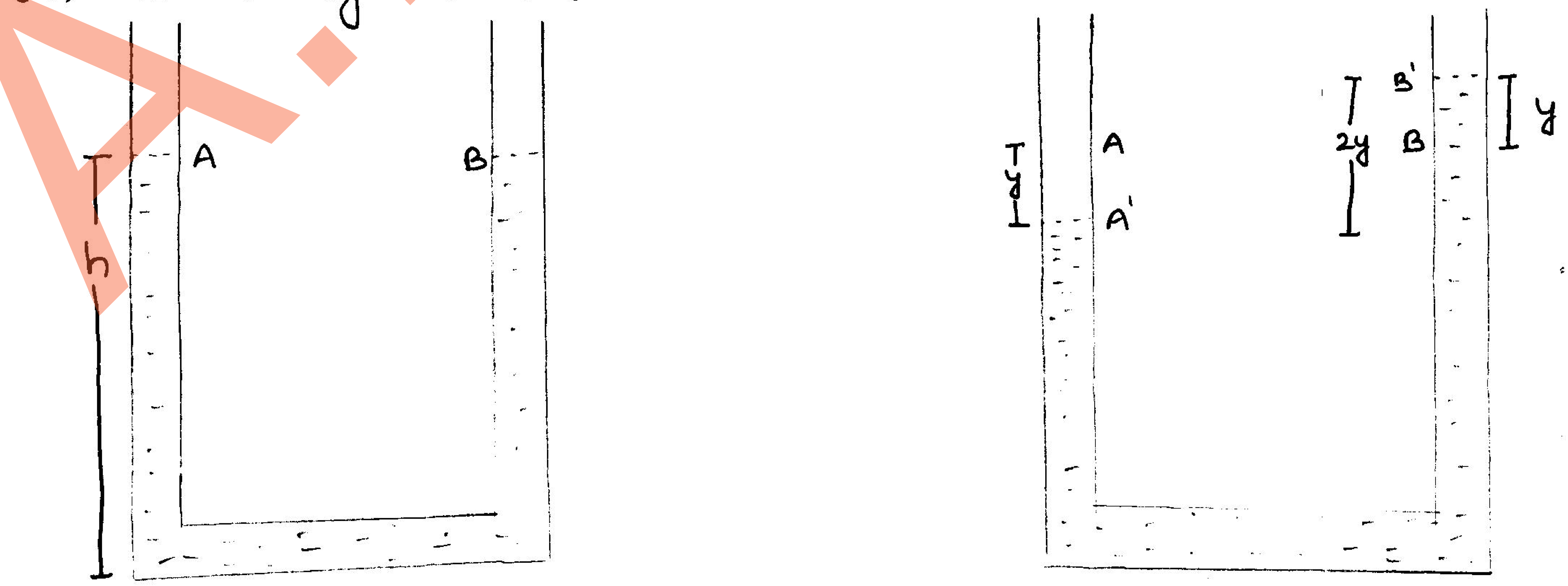
$$\boxed{L = 99.3\ \text{cm}}$$

__Drawbacks of simple pendulum__

1) Requirements can't be realised in actual practice.
2) Motion of bob is not strictly linear (may be rotating about point of suspension)
3) Suspension thread slackens when pendulum approaches extreme position

__Examples of S.H.M__

(a) Motion of a liquid column in U-tube



Consider a U-tube of uniform area 'A' & containing a liquid of density 'ρ' upto height 'h' in each limb.

Suppose liquid in left limb gets depressed through $AA' = y$ so liquid in right limb rise by $BB' = y$

The weight of liquid column of height '2y' provides the restoring force so as to make the limbs equal again.

$$F = -mg$$
$$= -\text{volume of liquid} \times \text{density} \times g$$
$$= -A \times 2y \times \rho \times g$$
$$= -2Ay\rho g$$

Let the horizontal position of U-tube is of negligible length, then the mass of the liquid is

$$m = \text{volume} \times \text{density}$$
$$= A(2h)\rho$$

If the restoring force produces an acceleration 'a' in the motion of liquid, then

$$a = \frac{F}{m}$$
$$= -\frac{2Ay\rho g}{2Ah\rho}$$
$$= -\frac{g}{h}y$$

g, h - constant, so, $\underline{a \propto y}$ & acts towards mean position

so motion is SHM

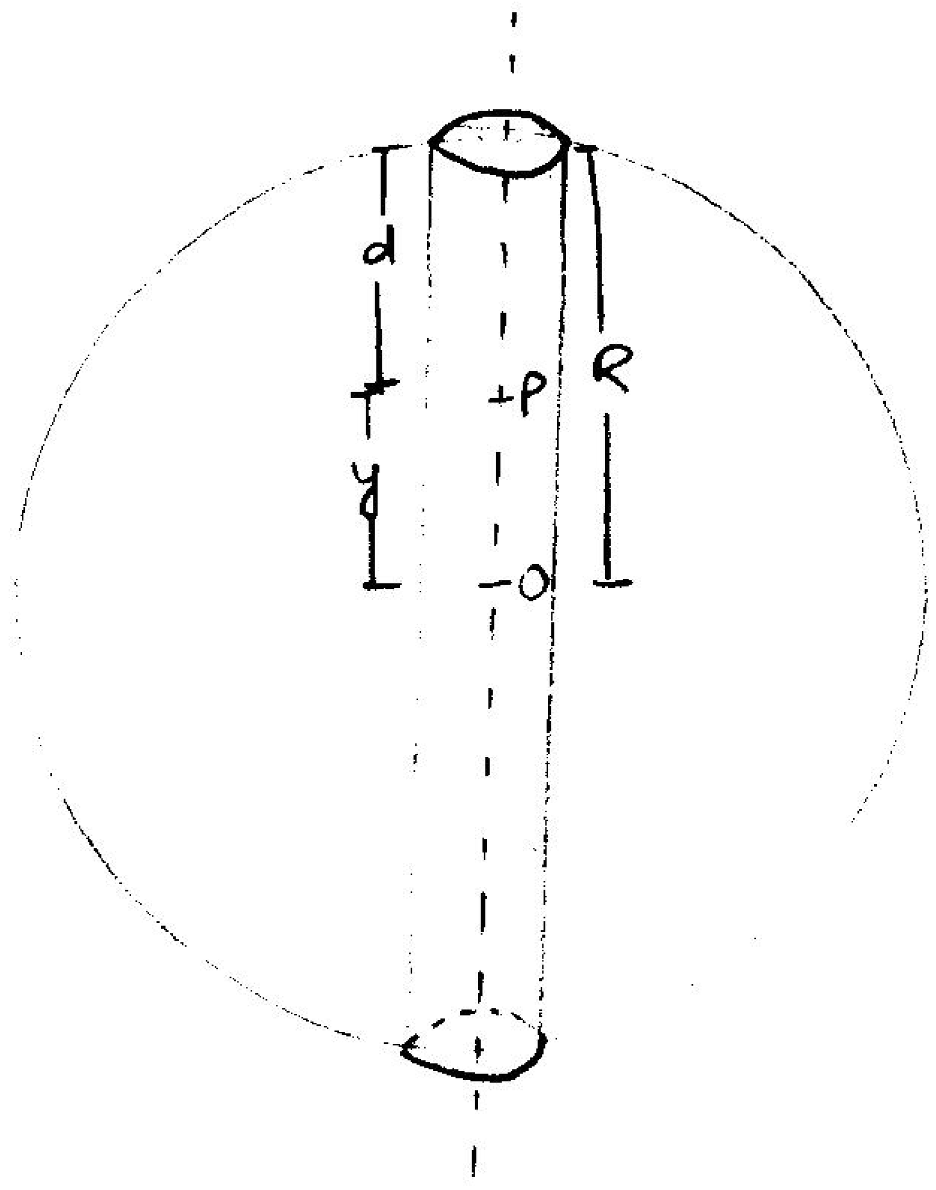
$$T = 2\pi\sqrt{\frac{y}{a}}$$
$$\boxed{T = 2\pi\sqrt{\frac{h}{g}}}$$

Moreover T is independent of

(i) Area of tube (A)

(ii) density of light (ρ)

(b) Motion of a body dropped in tunnel along the diameter of earth



Consider a body of mass 'm' is dropped into the tunnel & at any instant it is at point P.

Acceleration due to gravity at P is

$$g' = -g\left(1-\frac{d}{R}\right) \quad \left[\begin{array}{l}\text{-ve sign indicates that as 'd'} \\ \text{increases, } g' \text{ decreases}\end{array}\right]$$

$$= -g\frac{(R-d)}{R}$$

$$= -\frac{g}{R}y \qquad [\because\ y = R-d]$$

g, R - const. so, acc. $\propto y$ & is directed towards mean position (centre of earth), so the motion is SHM.

Now, $T = 2\pi\sqrt{\frac{y}{g'}}$

$$\boxed{T = 2\pi\sqrt{\frac{R}{g}}}$$

* T is independent of mass of the body.

Speed of transverse waves on a stretched string

The speed of transverse waves (v) on a string depends on

(i) mass per unit length of string (m),

(ii) tension in the string (T)

$$v \propto m^a T^b$$

$$v = K m^a T^b$$

$$[M^0LT^{-1}] = K\,[ML^{-1}]^a\,[MLT^{-2}]^b$$

$$[M^0LT^{-1}] = [M]^{a+b}\,[L]^{-a+b}\,[T]^{-2b}$$

Applying principle of homogenity

$$a+b = 0$$

$$-a+b = 1$$

$$-2b = -1$$

$$\therefore \quad a = -\frac{1}{2}, \quad b = \frac{1}{2}$$

So,

$$\boxed{v = \sqrt{\frac{T}{m}}}$$

Speed of longitudinal waves in a fluid

Speed of longitudinal waves (v) depends upon

(a) bulk modulus (B)

(b) density (ρ)

So,

$$v \propto B^a \rho^b$$

$$v = K B^a \rho^b$$

$$[LT^{-1}] = [ML^{-1}T^{-2}]^a\,[ML^{-3}]^b$$

$$[LT^{-1}] = [M]^{a+b}\,[L]^{-a-3b}\,[T]^{-2a}$$

Applying principle of homogenity

$$a+b=0$$
$$-a-3b=1$$
$$-2a=-1$$

$\therefore \quad a=\frac{1}{2}, \; b=-\frac{1}{2}$

$$\boxed{v=\sqrt{\frac{B}{\rho}}}$$

Newton's formula for velocity of sound in gases

$$v=\sqrt{\frac{B}{\rho}}$$

Newton assumed that the changes in pressure & volume of a gas, when sound waves propagate through it are isothermal.

[Amount of heat produced during compression is lost to surrounding
" " " lost " rarefaction " gained from " ,
so as to keep tem. const.]

$$\therefore \quad v=\sqrt{\frac{B_i}{\rho}}$$

Consider a gas having initial pressure P & volume V

For isothermal $\quad PV = \text{const.}$

Differentiating both sides

$$PdV + VdP = 0$$

$$P = -\frac{VdP}{dV} = B_i$$

$$\therefore \quad \boxed{v=\sqrt{\frac{P}{\rho}}}$$

At NTP, $P = 0.76 \times 13.6 \times 10^3 \times 9.8 \ Nm^{-2}$

$\rho = 1.293 \ kgm^{-3}$

$$v = \sqrt{\frac{P}{\rho}} = 280 \ ms^{-1}$$

Experimental value at NTP = $332 \ ms^{-1}$

Diff. = $332 - 280 = 52 \ ms^{-1}$

$$\% \text{ error} = \frac{52}{332} \times 100 = 16\%$$

As it was having a very large error (16%), it was concluded that there was an error in Newton's formula.

Laplace's correction

According to laplace, the changes in pressure & volume of a gas, when sound waves are propagated through it, are not isothermal but adiabatic because:

(i) The pulses of compression & rarefaction follow one another so rapidly that there is no exchange of heat among themselves or with the surrounding.

(ii) A gas is a bad conductor of heat so no exchange of heat betn system & surrounding.

$$\therefore \quad v = \sqrt{\frac{B_a}{\rho}}$$

Under adiabatic conditions

$$PV^\gamma = \text{const.}$$

$$P(\gamma V^{\gamma-1} dV) + V^\gamma dP = 0$$

$$\gamma P = \frac{-V^\gamma}{V^{\gamma-1}} \frac{dP}{dV} = B_a$$

$B_a = \gamma P$

$$\therefore \boxed{v = \sqrt{\frac{\gamma P}{\rho}}}$$

At NTP $\sqrt{\frac{P}{\rho}} = 280\ ms^{-1}$, $\gamma = 1.41$

So, $v = \sqrt{1.41} \times 280 = 332.5\ ms^{-1}$

As, Experimental & theoretical values are in agreement so Laplace's collection was valid.

Factors affecting velocity of sound

$$v = \sqrt{\frac{\gamma P}{\rho}}$$

(a) Effect of density

$$v \propto \frac{1}{\sqrt{\rho}}$$

(b) Effect of pressure

$$\rho = \frac{M}{V}$$

$$v = \sqrt{\frac{\gamma P V}{M}}$$

When T - const. , PV = const.

$\therefore \boxed{v = \text{const.}}$ (at const. tem.)

(c) Effect of tem.

$$PV = RT$$

$$P = \frac{RT}{V}$$

$$\therefore \quad v = \sqrt{\frac{\gamma RT}{\rho V}} = \sqrt{\frac{\gamma RT}{M}}$$

$$\boxed{v \propto \sqrt{T}}$$

Sound travel faster on a hot summer day than on a cold winter day.

(d) Effect of humidity

→ Presence of water vapours in air changes its density, so velocity of sound changes with humidity of air.

velocity of sound in dry air, $v_d = \sqrt{\frac{\gamma P}{\rho_d}}$

" " " moist " , $v_m = \sqrt{\frac{\gamma P}{\rho_m}}$

$$\frac{v_m}{v_d} = \sqrt{\frac{\rho_d}{\rho_m}}$$

Presence of water vapour reduces the density of air, $\rho_m < \rho_d$

$$\therefore \quad \boxed{v_m > v_d}$$

So, sound travels faster on a rainy day than on a dry day.