Moment of Inertia

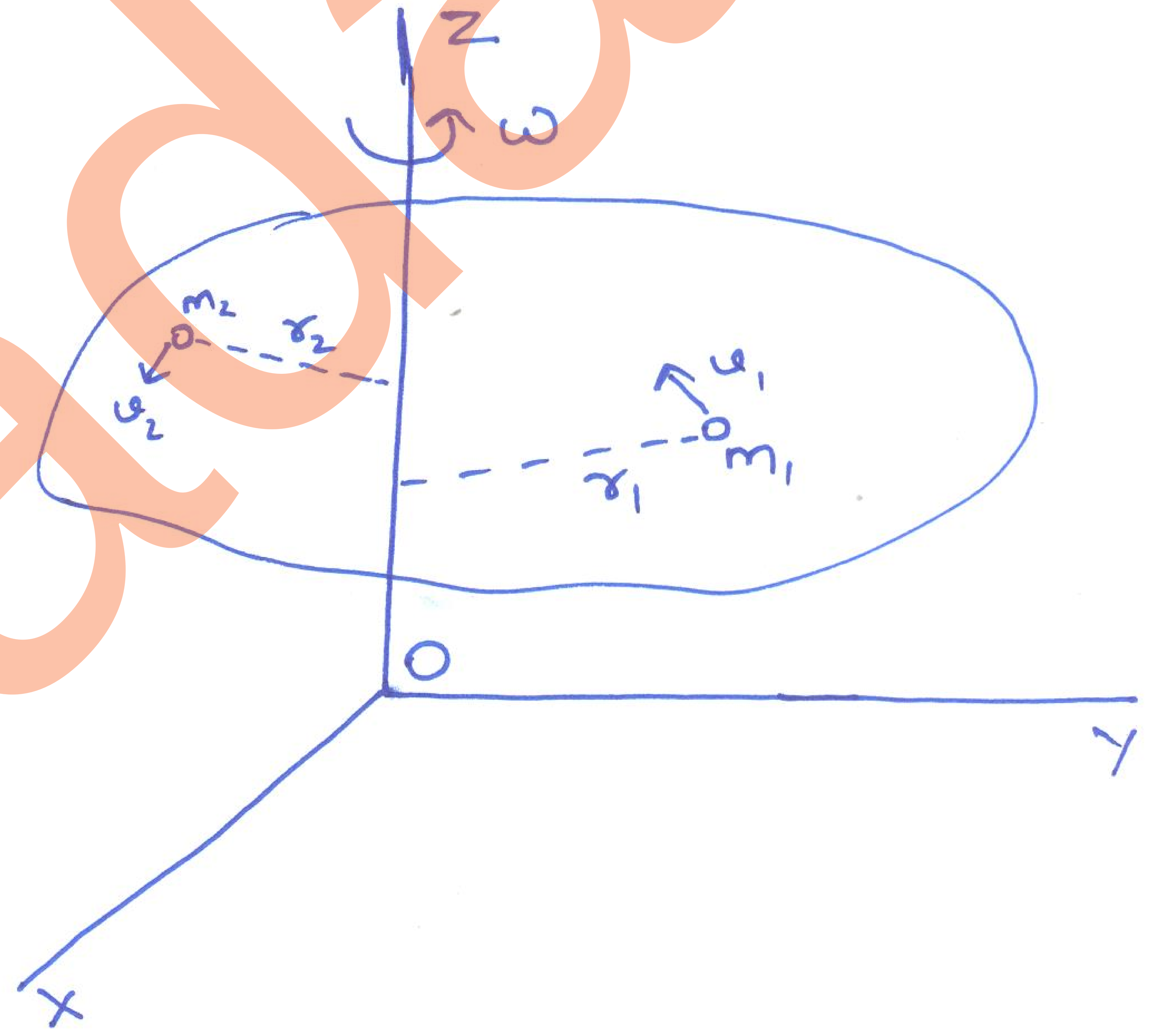
Moment of inertia

A quantity that measures the inertia of rotational motion of the body is called moment of inertia of the body.

* Moment of inertia is rotational analogue of mass in linear motion.

Kinetic energy of rotation

It is the energy possessed by the body on account of its rotation about a given axis.



Consider a body having particles of masses $m_1, m_2, ---$ at perpendicular distances $r_1, r_2 ---$ respectively from the axis of rotation.

The angular velocity ω of all the particles is same but the linear velocity will be different.

Kinetic energy of particle of mass $m_1 = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 r_1^2 \omega^2$

" " " " " " $m_2 = \frac{1}{2} m_2 r_2^2 \omega^2$ & so on

So, Kinetic energy of rotation of the body is

$$= \frac{1}{2} m_1 r_1^2 \omega^2 + \frac{1}{2} m_2 r_2^2 \omega^2 + \frac{1}{2} m_3 r_3^2 \omega^2 + - - - - - -$$

$$= \frac{1}{2} \left(\sum_{i=1}^{n} m_i r_i^2 \right) \omega^2$$

$$\boxed{\text{K.E. of rotation} = \frac{1}{2} I \omega^2}$$

where $I = \sum_{i=1}^{n} m_i r_i^2$ → Moment of inertia

∴ Moment of inertia of a body about a given axis is the sum of products of masses of all the particles of the body & squares of their respective perpendicular distances from the axis of rotation.

If $\omega = 1$

K.E. of rotation $= \frac{1}{2} I$

$$\boxed{I = 2 \times \text{K.E. of rotation}}$$

So, M. of. inertia of a body about a given axis is equal to twice the K.E. of rotation of the body rotating with unit angular velocity about the given axis.

Unit of I $\rightarrow$ $kg\,m^2$

Dimension $\rightarrow [ML^2]$

* Value of I depends upon –

(a) position of the axis of rotation

(b) shape & size of the body

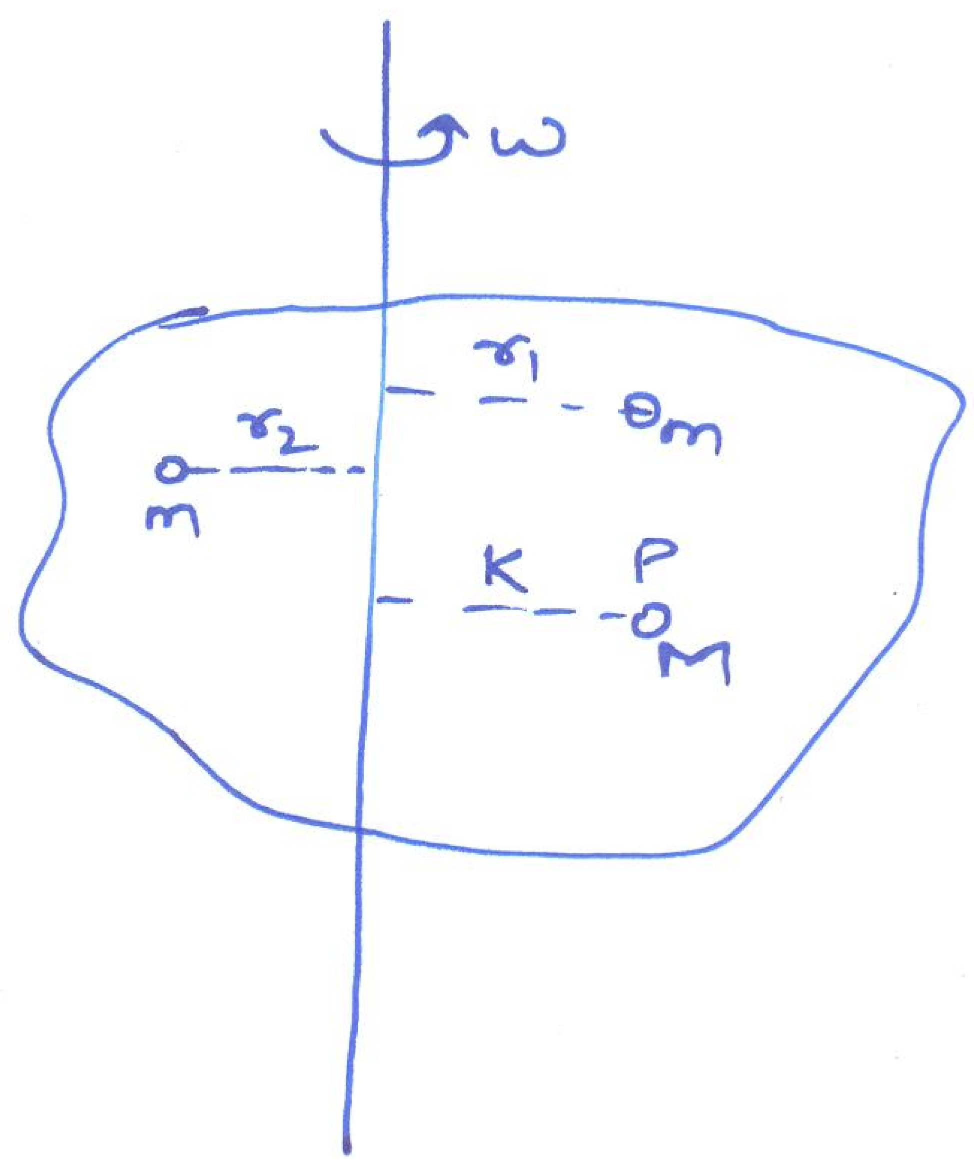
(c) distribution of mass of the body about the axis of rotation.

Radius of gyration (K)

The radius of gyration of a body about a given axis is the perpendicular distance of a pt. P from the axis, where if whole mass of the body were concentrated, the body shall have the same moment of inertia as it has with the actual distribution of mass.

Consider a rigid body of n particles each of m.

Let $r_1, r_2, \ldots$ be the $\perp^r$ distances of these particles from the axis of rotation.



$$I = mr_1^2 + mr_2^2 + \cdots + m_n r_n^2$$

$$= m(r_1^2 + r_2^2 + \cdots + r_n^2)$$

$$I = m \times n \left(\frac{r_1^2 + r_2^2 + \cdots + r_n^2}{n} \right)$$

$$= M \left(\frac{r_1^2 + r_2^2 + \cdots + r_n^2}{n} \right) \quad \text{—— (1)}$$

$\because M = m \times n$ — Total mass of the body

If the total mass of the body were concentrated at P at a $\perp^r$ distance K from the axis, then

$$I = MK^2 \quad \text{—— (2)}$$

from (1) & (2)

$$\boxed{K = \sqrt{\frac{r_1^2 + r_2^2 + \cdots + r_n^2}{n}}}$$

Unit of K – metre

<u>Value of K depends upon</u>

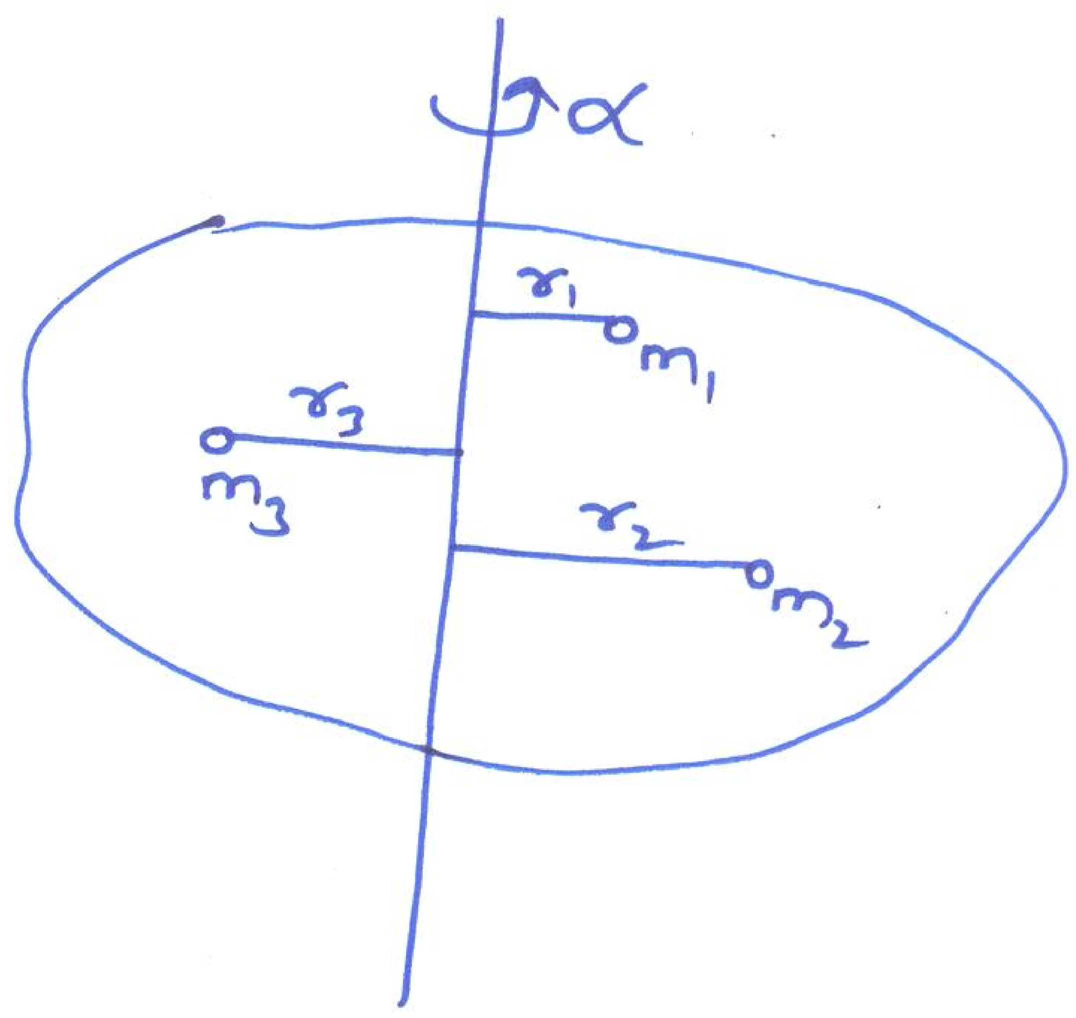
(a) shape & size of body

(b) position & configuration of the axis of rotation

(c) distribution of mass of body w.r.t. the axis of rotation

<u>Relation betn τ & I</u>

Suppose a rigid body consists of n particles of masses $m_1, m_2, \ldots m_n$ at $\perp^r$ distances $r_1, r_2, \ldots r_n$ resp. from the axis of rotation.



As the body is rigid, angular acceleration of all the particles of the body is same but their linear acc. are different.

Force on particle of mass m_1, $f_1 = m_1 a_1 = m_1 r_1 \alpha$

Moment of force f_1 about the axis of rotation, $\tau_1 = f_1 \times r_1$
$= m_1 r_1^2 \alpha$

" " " f_2 " " " " , $\tau_2 = m_2 r_2^2 \alpha$ & so on

$\therefore$ Torque acting on the body

$$\tau = m_1 r_1^2 \alpha + m_2 r_2^2 \alpha + \cdots\cdots + m_n r_n^2 \alpha$$

$$= (m_1 r_1^2 + m_2 r_2^2 + \cdots\cdots + m_n r_n^2)\alpha$$

$$= \left(\sum_{i=1}^{n} m_i r_i^2\right)\alpha$$

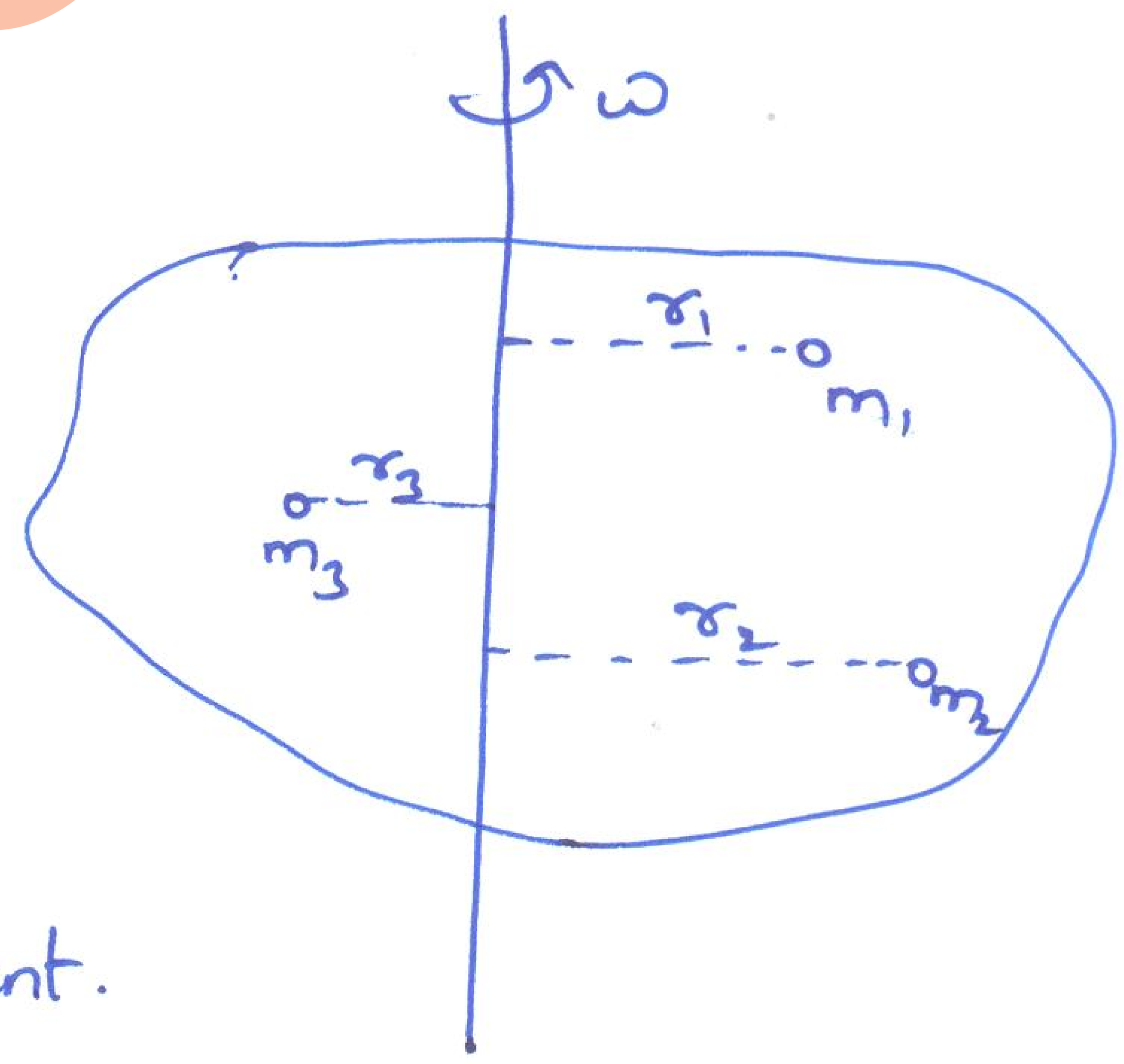
$$\boxed{\tau = I\alpha}$$

If $\alpha = 1$ $\boxed{I = \tau}$

So, moment of inertia of a body about a given axis is numerically equal to torque acting on the body rotating with unit angular acc. about it.

Relation betn L & I

Suppose a rigid body consists of n particles of masses $m_1, m_2, \cdots m_n$ at $\perp^r$ distances $r_1, r_2, \cdots r_n$ resp. from the axis of rotation.



As the body is rigid, angular velocity ω of all the particles of the body is same but the linear velocities are different.

Linear momentum of particle of mass m_1, $p_1 = m_1 v_1 = m_1 r_1 \omega$

Angular momentum of this particle about given axis, $L_1 = p_1 \times r_1$

$$= m_1 r_1 \omega \times r_1$$

$$= m_1 r_1^2 \omega$$

" " of m_2 " " " , $L_2 = m_2 r_2^2 \omega$ & so on

$\therefore$ Angular momentum of the body

$$L = m_1 r_1^2 \omega + m_2 r_2^2 \omega + \cdots\cdots + m_n r_n^2 \omega$$

$$= \left(\sum_{i=1}^{n} m_i r_i^2\right)\omega$$

$$\boxed{L = I\omega}$$

If $\omega = 1$, $\boxed{I = L}$

So, moment of inertia of a body about a given axis is numerically equal to angular momentum of the body rotating with unit angular velocity about that axis.

Relation betn τ & L

$$L = I\omega$$

$$\frac{dL}{dt} = I\left(\frac{d\omega}{dt}\right) = I\alpha \quad \text{—— (1)}$$

$$\boxed{\tau = I\alpha} \quad \text{—— (2)}$$

from (1) & (2)

$$\boxed{\tau = \frac{dL}{dt}}$$

Principle of conservation of angular momentum

" When no external torque acts on a system of particles, then the total angular momentum of the system remains always a constant. "

As, $\tau = \dfrac{dL}{dt}$

if $\tau = 0$, $\dfrac{dL}{dt} = 0$

$$\boxed{L = \text{constant}}$$

Examples of Law of conservation of angular momentum

(1) The angular velocity of revolution of a planet around the sun in an elliptical orbit increases, when the planet comes closer to the sun and vice-versa.

Reason :

When a planet comes closer to sun, I decreases.

As $L = I\omega$ so if I is decreasing so ω will increase.

(2) A circus acrobat performs feats involving spin by bringing her arms & legs closer to her body or vice-versa.

Reason:

On bringing the arms & legs closer to the body, I decreases

As $L = I\cdot\omega$, so if I is decreasing so ω will increase.

(3) While jumping, a cat stretches its body alongwith its tail so I increases, so ω decreases & the cat lands gently on its feet.

(4) Neutron star spins rapidly

Reason: After all the fuel is used up, star is an isolated system which starts shrinking.

Due to shrinking, I decreases so ω increases.

Q. The plane of the orbit of a planet can never change on its own. Why?

Ans → No external τ acting → L constant (in mag. & direction)

So, orbit of every planet will be in a fixed plane.

Theorems on moment of inertia

(1) Theorem of parallel axes

"Moment of inertia of a rigid body about any axis AB is equal to moment of inertia of the body about another axis KL passing through centre of mass C of the body in a direction parallel to AB, plus the product of total mass M of the body and square of the perpendicular distance betⁿ the 2 parallel axes AB and KL."

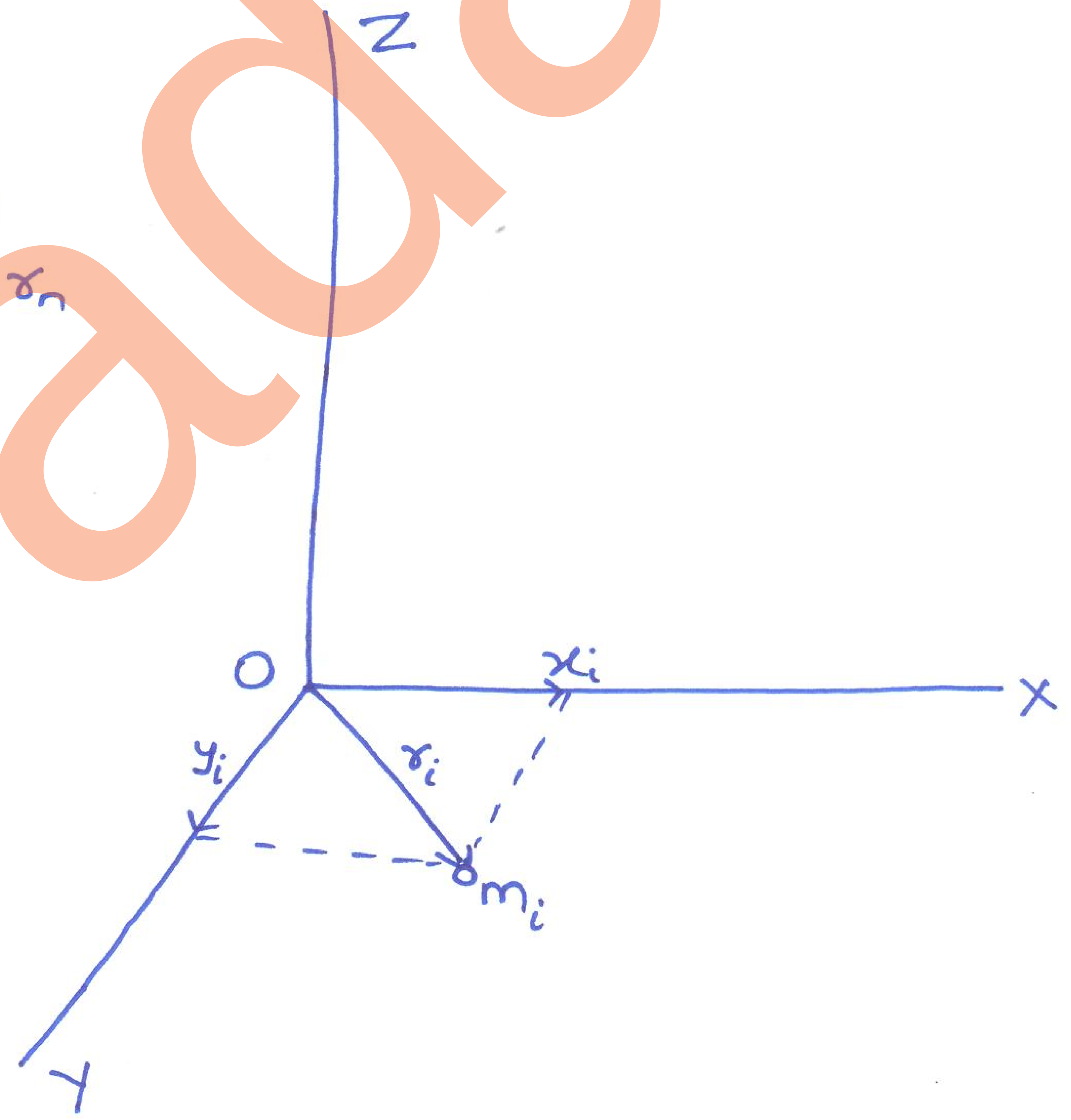
$$\boxed{I_{AB} = I_{KL} + Mh^2}$$

② Theorem of perpendicular axis

"Moment of inertia of a plane lamina about any axis OZ perpendicular to the plane of the lamina is equal to sum of the moments of inertia of the lamina about any two mutually perpendicular axes OX & OY in the plane of the lamina, meeting at a point where the given axis OZ passes through the lamina."

$$\boxed{I_z = I_x + I_y}$$

Proof:



Consider a lamina having n particles of masses $m_1, m_2 \ldots m_n$ at perpendicular distances $r_1, r_2 \ldots r_n$ from OZ.

Let $x_1, x_2 \ldots x_n$ – ⊥r distances of these particles from OY

$y_1, y_2 \ldots y_n$ – ⊥r distances of these particles from OX

$$\therefore \quad I_x = m_1 y_1^2 + m_2 y_2^2 + \ldots + m_n y_n^2 = \sum_{i=1}^{n} m_i y_i^2$$

$$I_y = m_1 x_1^2 + m_2 x_2^2 + \ldots + m_n x_n^2 = \sum_{i=1}^{n} m_i x_i^2$$

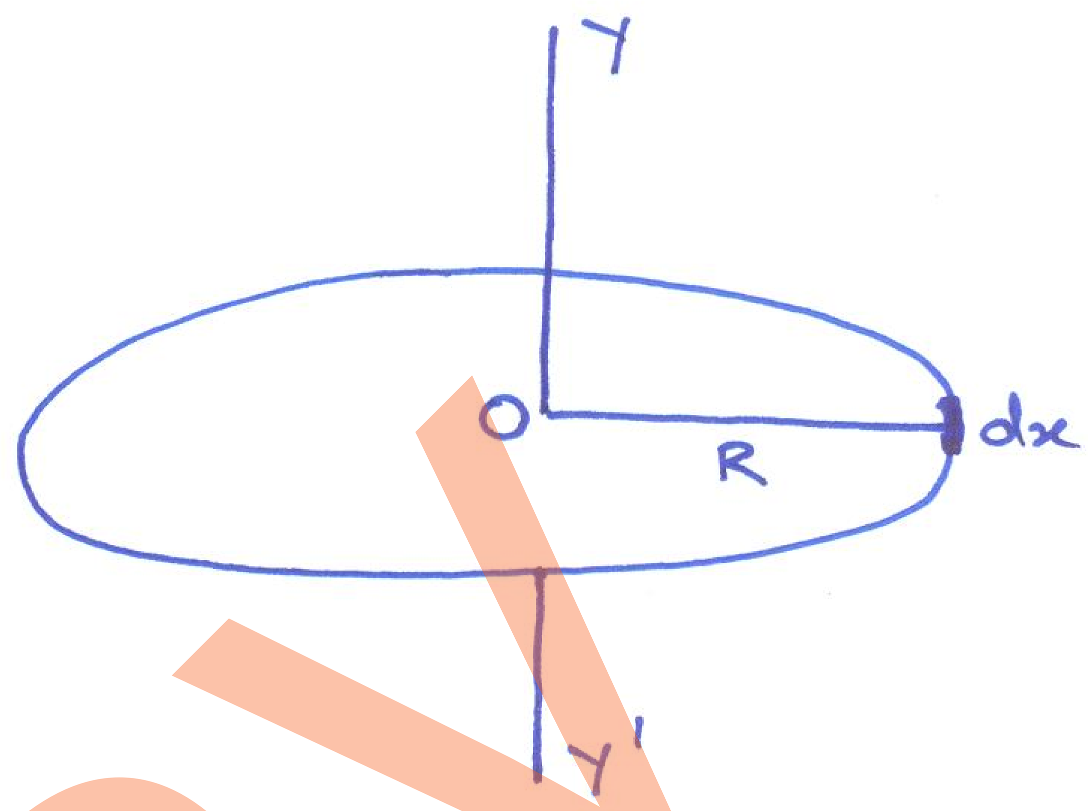
$$I_z = m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2 = \sum_{i=1}^{n} m_i r_i^2$$

$$I_x + I_y = \sum_{i=1}^{n} m_i y_i^2 + \sum_{i=1}^{n} m_i x_i^2 = \sum_{i=1}^{n} m_i (x_i^2 + y_i^2) = \sum_{i=1}^{n} m_i r_i^2$$

$$\therefore \quad \boxed{I_x + I_y = I_z}$$

Moment of inertia of a thin circular ring

(i) about an axis (YOY'), ⊥r to the plane of ring & passing through its centre.



Consider a thin circular ring of mass M & radius R with centre O.

Length of the ring $= 2\pi R$

Mass per unit length of ring $= M/2\pi R$

Consider a small element of the ring of length 'dx'.

Mass of element 'dx' $= \frac{M}{2\pi R} dx$

Moment of inertia of element 'dx' about YOY' $= \left(\frac{M}{2\pi R} dx\right) R^2$

$$= \frac{MR}{2\pi} dx$$

Moment of inertia of entire ring about YOY'

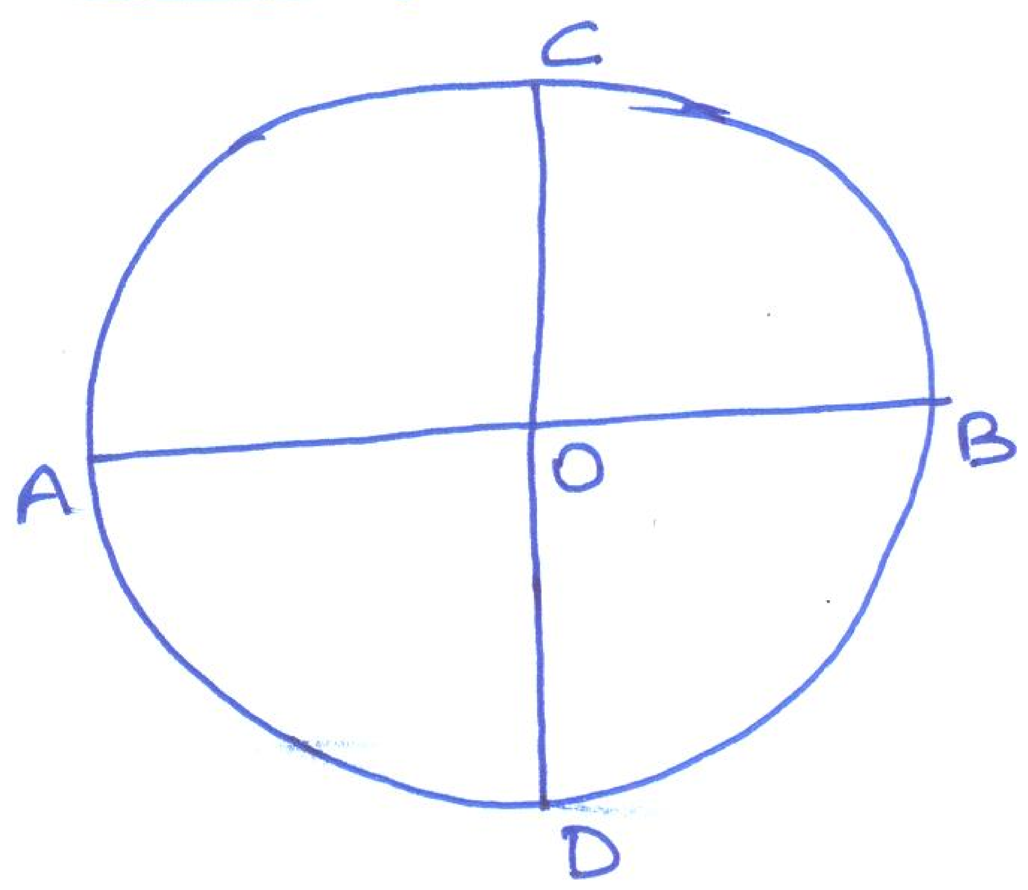
$$I = \int_0^{2\pi R} \frac{MR}{2\pi} dx = \frac{MR}{2\pi} \int_0^{2\pi R} dx$$

$$= \frac{MR}{2\pi} \left[x\right]_0^{2\pi R}$$

$$= \frac{MR}{2\pi} \left[2\pi R - 0\right]$$

$$\boxed{I = MR^2}$$

(ii) Moment of inertia of a uniform circular ring about any diameter of the ring.



Acc. to theorem of perpendicular axes

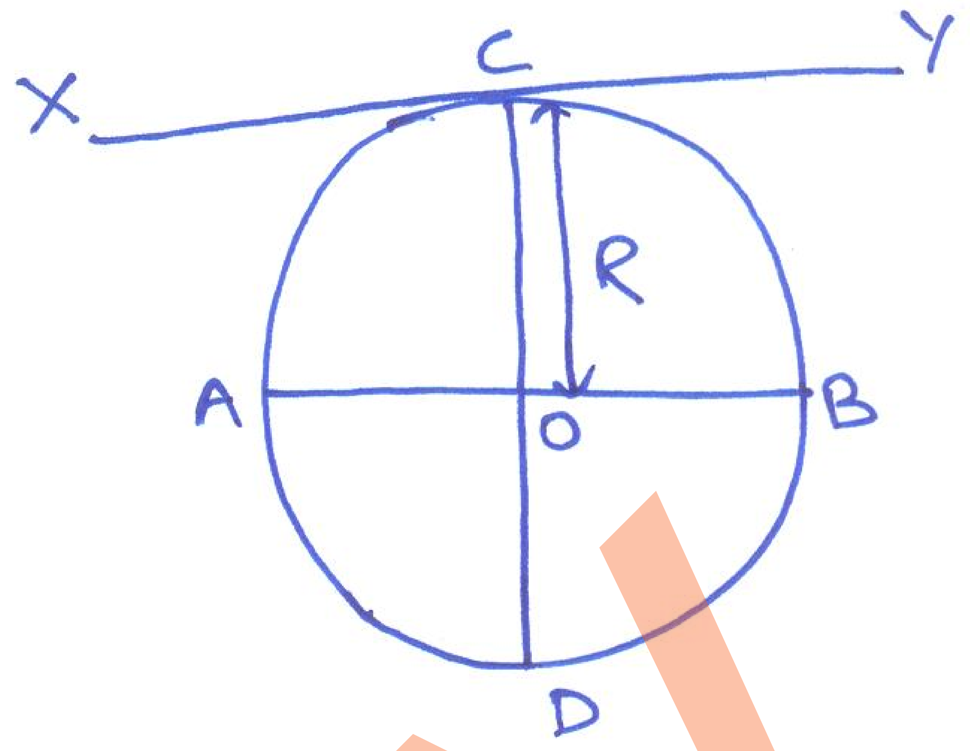
$$I_{AB} + I_{CO} = MR^2$$

$$I_d + I_d = MR^2$$

$$\boxed{I_d = \frac{1}{2} MR^2}$$

(iii) Moment of inertia of a uniform circular ring about a tangent in the plane of the ring.

Acc. to theorem of parallel axes



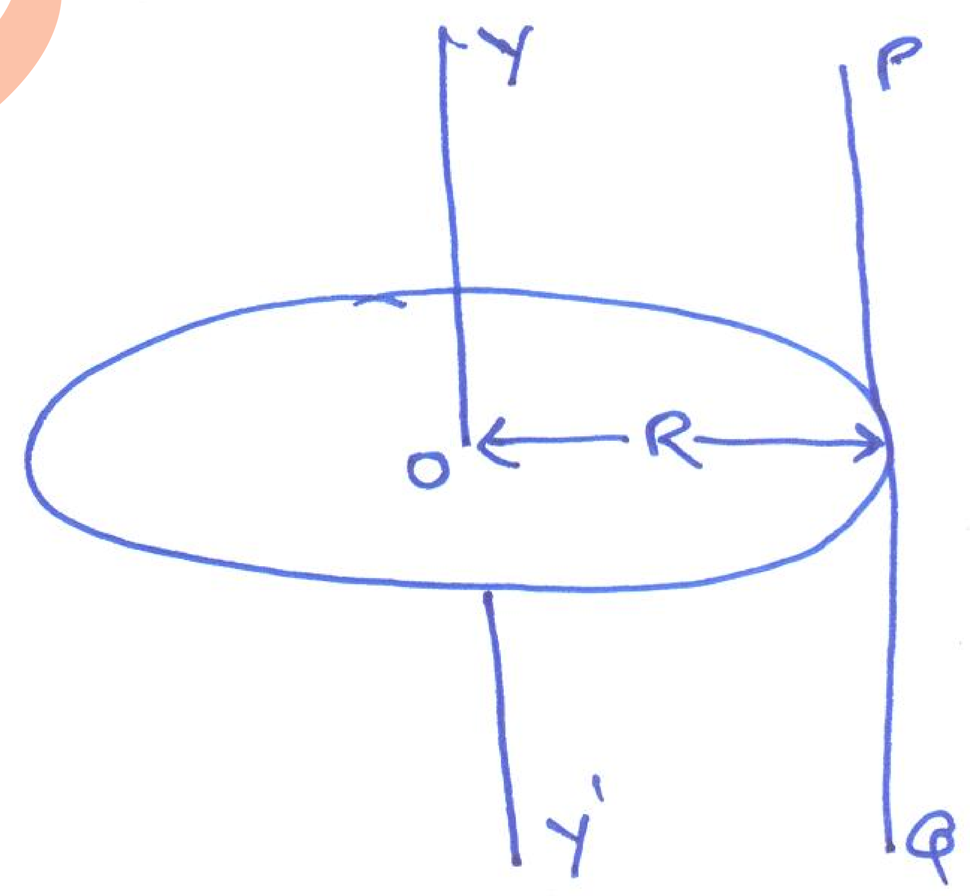
$$I_{XY} = I_{AB} + MR^2$$

$$= \frac{1}{2}MR^2 + MR^2$$

$$\boxed{I_{XY} = \frac{3}{2}MR^2}$$

(iv) Moment of inertia of a uniform circular ring about a tangent perpendicular to the plane of the ring

Acc. to theorem of parallel axes



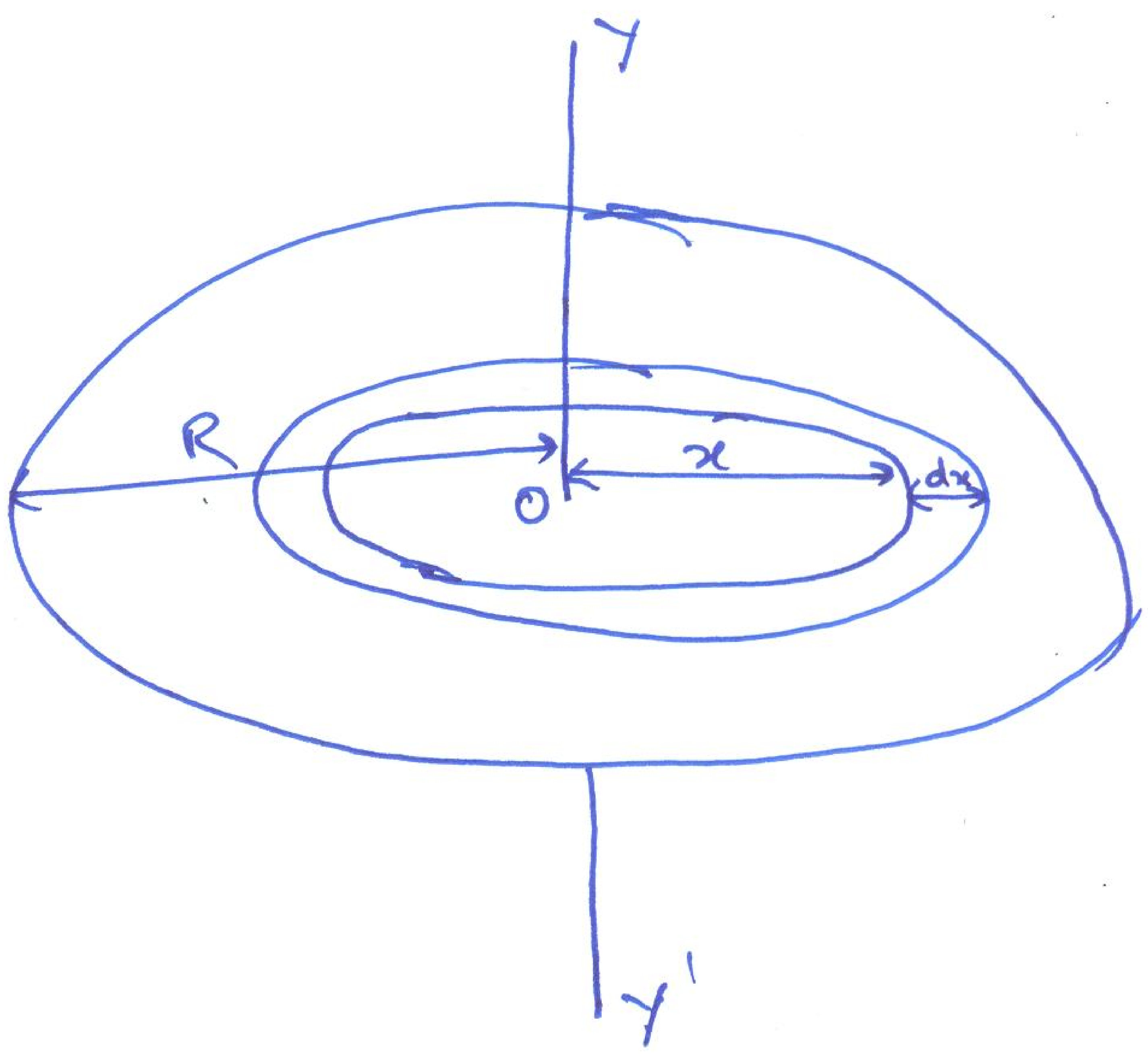
$$I_{PQ} = I_{YY'} + MR^2$$

$$= MR^2 + MR^2$$

$$\boxed{I_{PQ} = 2MR^2}$$

Moment of inertia of a uniform circular disc about an axis YOY' perpendicular to the plane of the disc and passing through the centre.

Consider a uniform circular disc of mass M & radius R with centre O.



Surface area of disc $= \pi R^2$

Mass per unit area of disc $= \frac{M}{\pi R^2}$

Consider a small element of disc of radius x & width dx

Length of element $= 2\pi x$

Surface area of element $= 2\pi x\, dx$

$\therefore$ Mass of element $= \dfrac{M}{\pi R^2}(2\pi x\, dx) = \dfrac{2Mx\, dx}{R^2}$

Moment of inertia of the element of the disc about YOY'

$$= \frac{2Mx\, dx}{R^2}\, x^2$$

$$= \frac{2Mx^3 dx}{R^2}$$

$\therefore$ Moment of inertia of the circular disc about YOY'

$$I = \int_0^R \frac{2Mx^3 dx}{R^2}$$

$$= \frac{2M}{R^2}\int_0^R x^3 dx$$

$$= \frac{2M}{R^2}\left[\frac{x^4}{4}\right]_0^R$$

$$= \frac{2M}{4R^2}\left[R^4 - 0\right]$$

$$= \frac{\cancel{2}M}{{}_2\cancel{4}\cancel{R^2}}\, \cancel{R^4}\, R^2$$

$$\boxed{I = \frac{1}{2}MR^2}$$