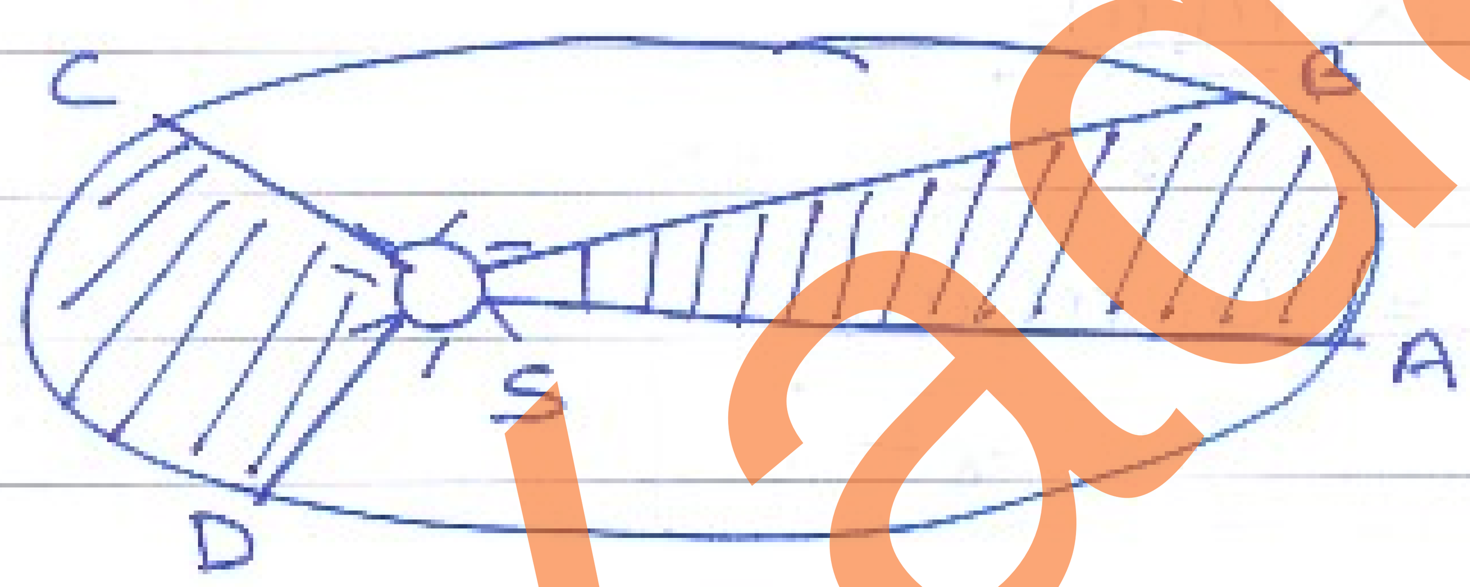
Gravitation

Kepler's Laws

1. **Law of orbit** - Each planet revolves around the sun in an elliptical orbit, with sun at one focus of the elliptical path.

2. **Law of area** - The position vector of the planet from the sun (i.e. the line joining the planet to the sun) sweeps out equal areas in equal intervals of time.



If time bet^n A to B & C to D is same then
area ABS = area CDS

3. **Law of periods** - The square of the time period of any planet about the sun is proportional to the cube of the semi-major axis of elliptical orbit

$$T^2 \propto r^3$$

or

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3}$$

Newton's law of gravitation

Every body in the universe attracts every other body with a force which is directly proportional to the product of their masses & inversely proportional to the square of the distance between them.

Consider 2 bodies of masses m_1 & m_2 lying at a distance r apart.

Acc. to law of gravitation

$$F \propto m_1 m_2$$

$$F \propto \frac{1}{r^2}$$

So,
$$F \propto \frac{m_1 m_2}{r^2}$$

$$\boxed{F = \frac{G\, m_1 m_2}{r^2}}$$

$G = 6.67 \times 10^{-11}\ Nm^2 kg^{-2}$
↳ universal gravitational constant.

Acceleration due to gravity (g)

The acceleration produced in the motion of a body under the effect of gravity is called acceleration due to gravity.

Consider an object of mass 'm' lying on the surface of earth.

Acc. to law of gravitation, the force of attraction on the object due to earth is

$$F = \frac{GMm}{R^2}$$

where M - mass of earth
R - radius " "

This force produces acceleration in the body

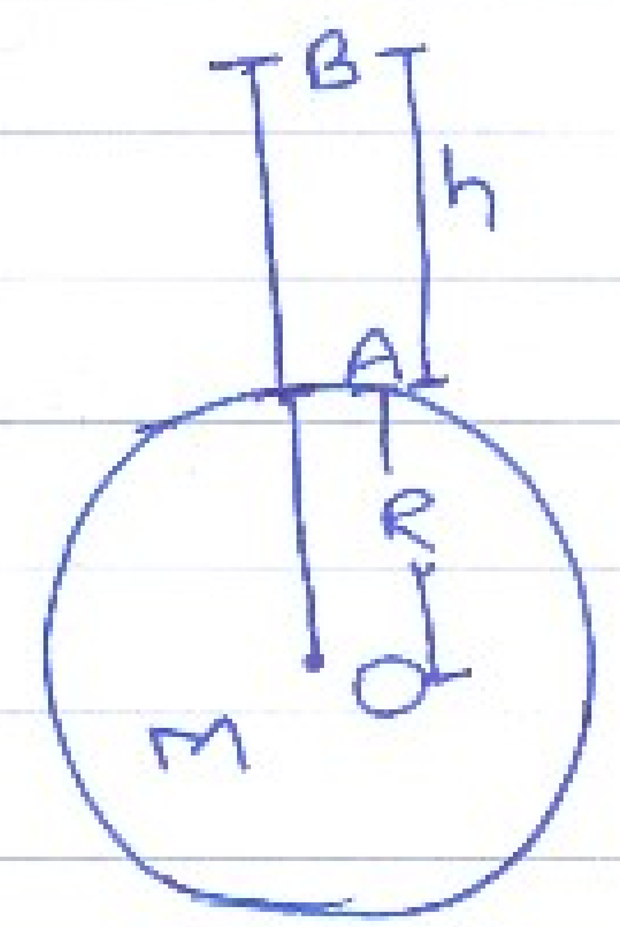
$$F = ma$$

So, $$ma = \frac{GMm}{R^2}$$

$$\boxed{g = \frac{GM}{R^2}} \qquad [\because a = g]$$

Variation of g

(a) Effect of altitude



At point A

$$g = \frac{GM}{R^2}$$

At pt. B

$$g' = \frac{GM}{(R+h)^2}$$

$$\frac{g'}{g} = \frac{\cancel{GM}}{(R+h)^2} \times \frac{R^2}{\cancel{GM}}$$

$$= \frac{R^2}{R^2\left(1+\frac{h}{R}\right)^2}$$

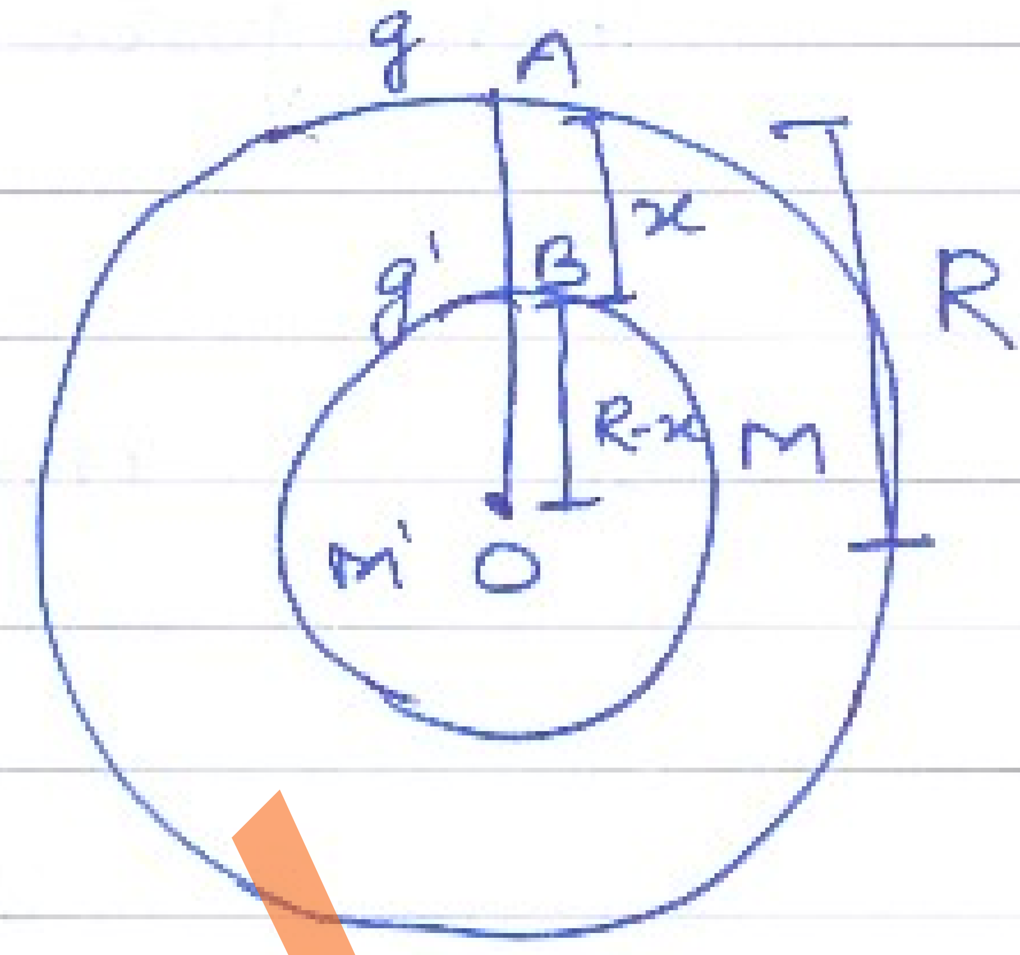
$$\boxed{\frac{g'}{g} = \left(1+\frac{h}{R}\right)^{-2}}$$ — Use this formula

$$\frac{g'}{g} = 1 - \frac{2h}{R} + \ldots$$ [Using Binomial theorem]

$$\boxed{\frac{g'}{g} = 1 - \frac{2h}{R}}$$ → Use this for small values of h

$g' < g$ it means value of g decreases with height.

(b) Effect of depth



At pt. A

$$g = \frac{GM}{R^2}$$

If ρ is density of the material of earth

$$\rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3}$$

$$M = \frac{4}{3}\pi R^3 \rho$$

So, $$g = \frac{G \times \frac{4}{3}\pi R^3 \rho}{R^2} = \frac{4}{3}\pi G R \rho$$

At point B

$$g' = \frac{GM'}{(R-x)^2}$$

Now, $$M' = \frac{4}{3}\pi (R-x)^3 \rho$$

So, $$g' = \frac{G \times \frac{4}{3}\pi (R-x)^3 \rho}{(R-x)^2} = \frac{4}{3}\pi G (R-x)\rho$$

$$\frac{g'}{g} = \frac{4/3\, G (R-x)\rho}{4/3\, G R \rho} = \frac{R-x}{R}$$

$$\boxed{\frac{g'}{g} = 1 - \frac{x}{R}}$$

$g' < g$ so value of g decreases with depth

* At centre of earth $x = R$, $g' = 0$

Gravitational Field

The space around a material body in which its gravitational force of attraction can be measured is called its gravitational field.

Intensity of gravitational field ($\vec{E}$)

The intensity of gravitational field of a body at any point in its field is defined as the force experienced by a unit mass placed at that point.

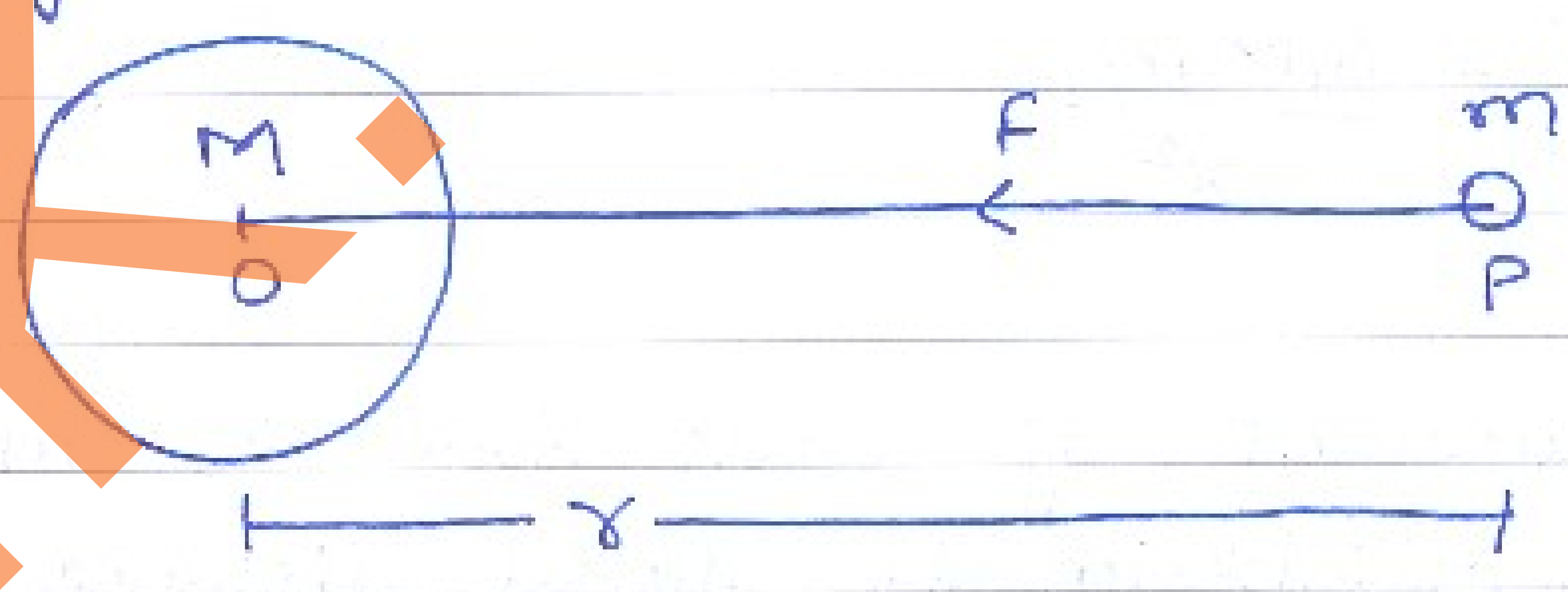
S.I unit → $N kg^{-1}$

cgs → $dyne\ g^{-1}$

$$\vec{E} = \frac{F}{m}$$

Intensity of gravitational field due to earth

Consider a point P at a distance 'r' from the centre of earth.



Force on test mass 'm' at pt. P is

$$F = \frac{GMm}{r^2}$$

Gravitational field of earth at pt. P is

$$\vec{E} = \frac{F}{m} = \frac{GMm}{r^2} \times \frac{1}{m}$$

$$\boxed{\vec{E} = \frac{GM}{r^2}}$$

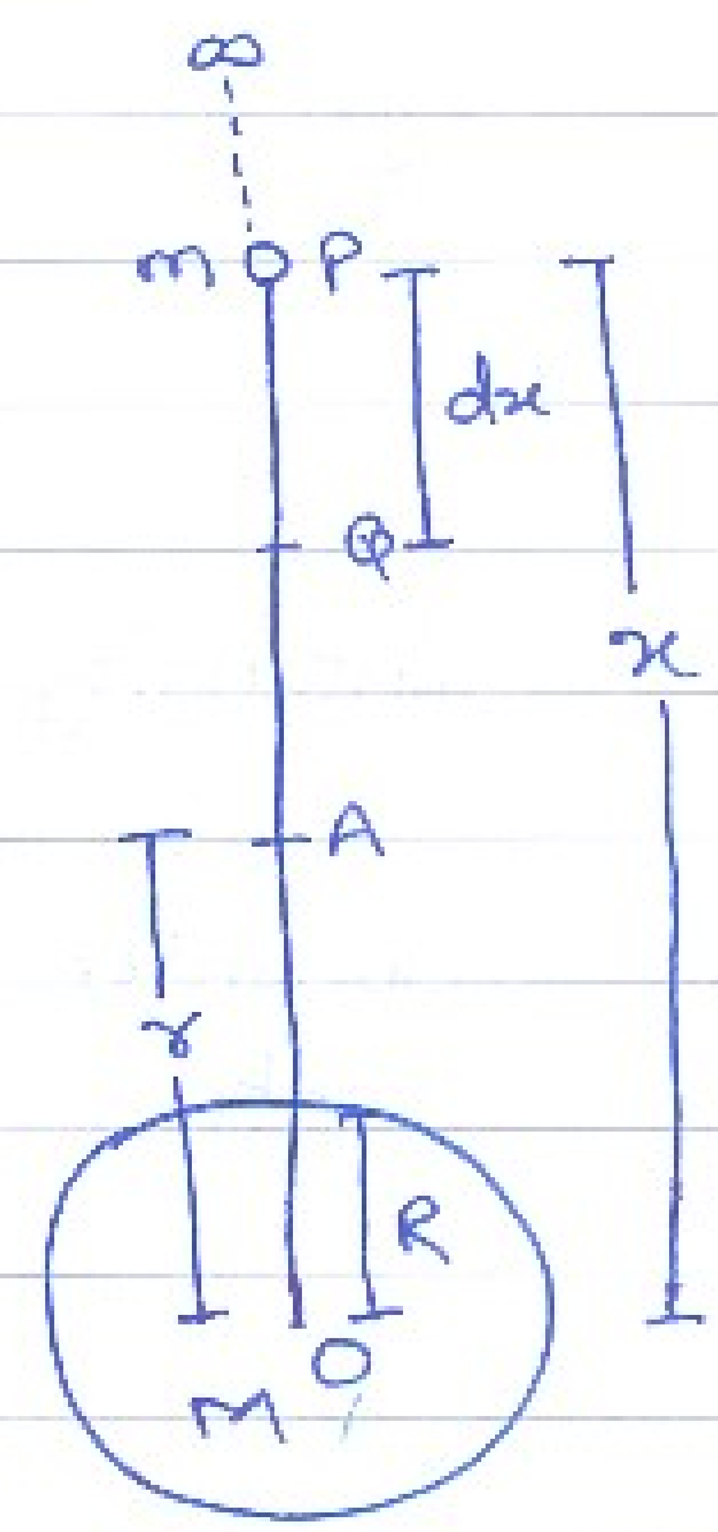
Now, $g = \frac{GM}{r^2}$

So, $\boxed{\vec{E} = g}$

Gravitational potential energy

Gravitational potential energy of a body at a point is defined as the amount of work done in bringing the body from infinity to that point against the gravitational force.

Consider a body of mass m lying ~~at~~ pt. A.



Suppose that at any instant, the body is at pt. P, then gravitional force on the body at pt. P is

$$F = \frac{GMm}{x^2}$$

Small amount of work done in moving the body through infinitesimally small distance PQ is

$$dW = F\,dx$$
$$= \frac{GMm}{x^2}\,dx$$

Work done in bringing the body from infinity to pt. A is

$$W = \int_{\infty}^{r} \frac{GMm}{x^2}\,dx$$

$$W = GMm \int_{\infty}^{r} x^{-2} dx$$

$$= GMm \left[\frac{x^{-1}}{-1} \right]_{\infty}^{r}$$

$$= -GMm \left[\frac{1}{r} - \frac{1}{\infty} \right]$$

$$W = -\frac{GMm}{r}$$

This work done is stored inside the body as its gravitational potential energy so

$$\boxed{U = -\frac{GMm}{r}}$$

Discussion

1. The negative sign shows that "to bring the body from ∞ to a distance 'r', work is done by gravitational field of earth.
2. As r increases, U becomes less $-ve$
   at $r = \infty$; $U = 0$ (max.)
3. If a body of mass 'm' is moved from a pt. at distance r_1 to a pt. at distance r_2 ($r_1 > r_2$) then change in potential energy is

$$\Delta U = \int_{r_1}^{r_2} \frac{GMm}{x^2} dx$$

$$= GMm \left[\frac{1}{r_1} - \frac{1}{r_2} \right]$$

→ If $r_1 > r_2$ $\Delta U = -ve$
$r_1 < r_2$ $\Delta U = +ve$ [formula same]

4. If $r_1 = R$ & $r_2 = R+h$

$$\Delta U = GMm\left[\frac{1}{R} - \frac{1}{R+h}\right]$$

$$= \frac{GMm}{R}\left[1 - \frac{R}{R+h}\right]$$

$$= \frac{GMm}{R}\left[1 - \frac{1}{\left(1+\frac{h}{R}\right)}\right]$$

$$= \frac{GMm}{R}\left[1 - \left(1+\frac{h}{R}\right)^{-1}\right]$$

$$= \frac{GMm}{R}\left[1 - \left(1 - \frac{h}{R} + \frac{h^2}{R^2} + \cdots\right)\right]$$

$$= \frac{GMm}{R}\left[1 - 1 + \frac{h}{R}\right]$$

$$= \frac{GMm}{R} \times \frac{h}{R}$$

$$= \frac{GM}{R^2} mh$$

$$\boxed{\Delta U = mgh}$$

<u>Gravitational potential</u>

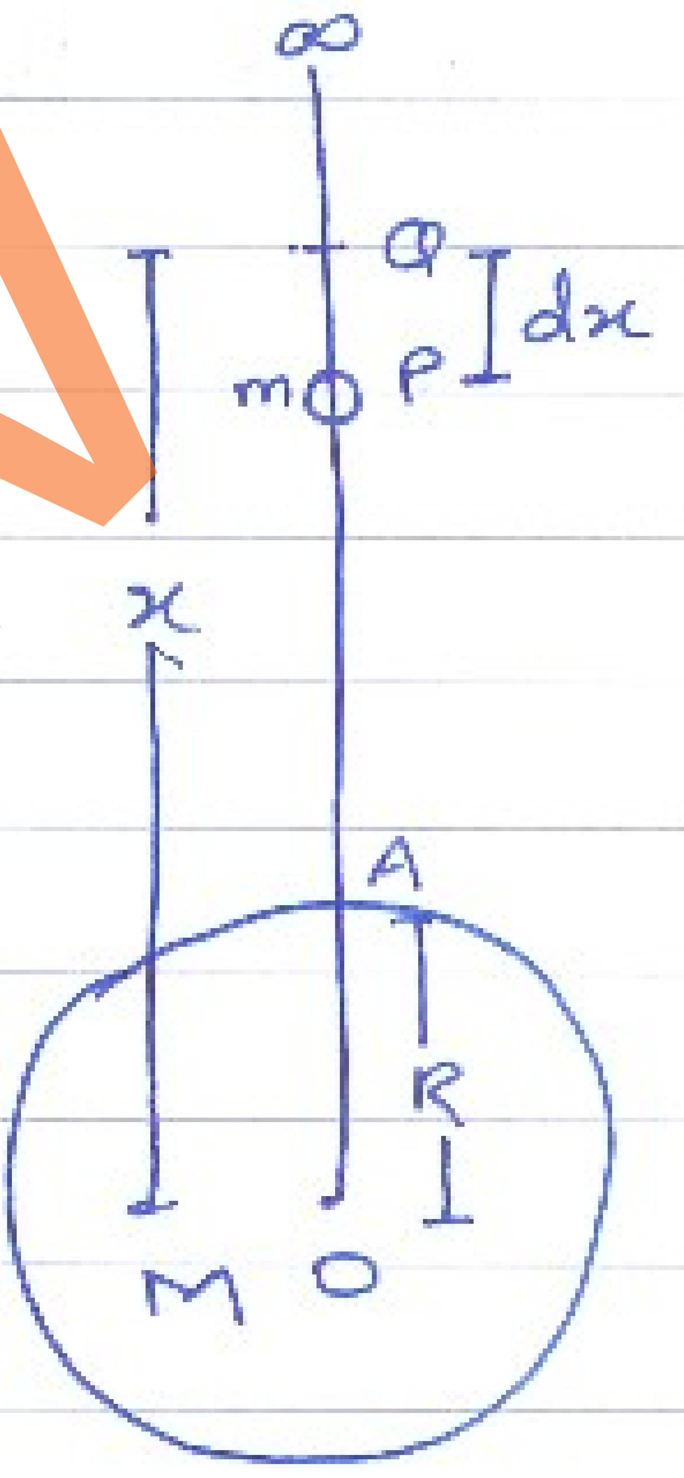
The gravitational potential at a pt. in the gravitational field of earth is defined as the amount of work done in bringing a body of unit mass from infinity to that point.

$$V = -\frac{GM}{r} \qquad \left[V = \frac{W}{m}\right]$$

Unit $-$ $J.kg^{-1}$

Escape velocity

The minimum velocity with which a body must be projected up so as to enable it to just overcome the gravitational pull, is known as escape velocity.



Consider a body of mass m at pt. P at a distance 'x' from the centre of earth.

Force exerted by earth on the body is

$$F = \frac{GMm}{x^2}$$

Work done in moving the body against gravitational attraction through dx is

$$dW = F\,dx$$

$$= \frac{GMm}{x^2}\,dx$$

Total work done in moving the body from surface of earth to infinity is

$$W = \int_R^\infty \frac{GMm}{x^2}\,dx$$

$$= GMm \int_R^\infty x^{-2}\,dx$$

$$= GMm \left[\frac{x^{-1}}{-1}\right]_R^\infty$$

$$= -GMm \left[\frac{1}{x}\right]_R^\infty$$

$$W = -GMm\left[\frac{1}{\infty} - \frac{1}{R}\right] = \frac{GMm}{R}$$

This work done is performed on the body by providing an equal amount of kinetic energy to it at the surface of earth.

$$\frac{1}{2} m v_e^2 = \frac{GMm}{R}$$

$$\boxed{v_e = \sqrt{\frac{2GM}{R}}}$$

$$v_e = \sqrt{\frac{2gR^2}{R}} \qquad [\because GM = gR^2]$$

$$\boxed{v_e = \sqrt{2gR}}$$

On putting $g = 9.8\ ms^{-2} = 0.0098\ kms^{-2}$

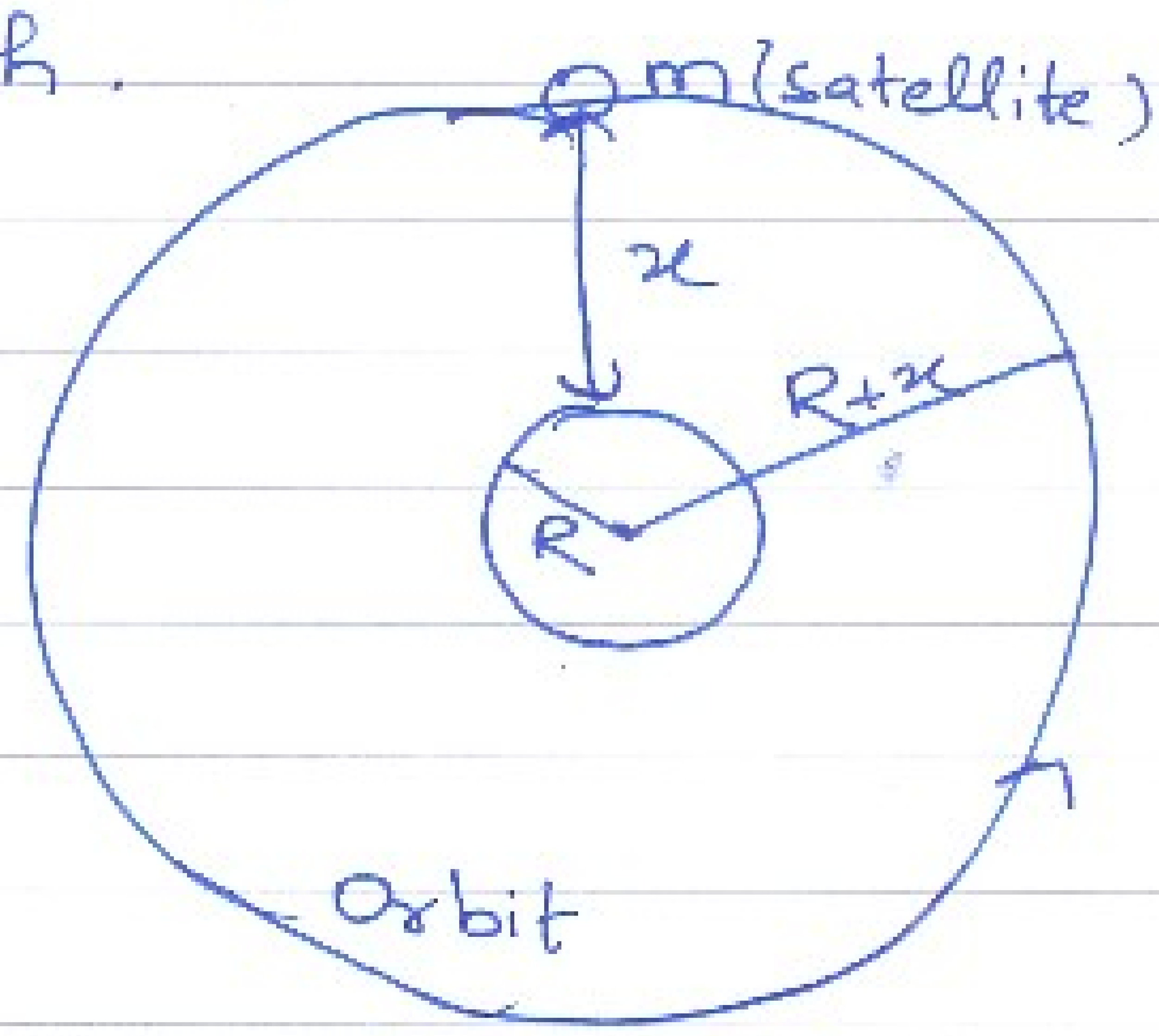
$R = 6400\ km$

$$\boxed{v_e = 11.2\ kms^{-1}}$$

Orbital velocity of a satellite

It is the velocity required to put the satellite into its orbit around the earth.

Consider a satellite of mass 'm' at a distance 'x' above the surface of earth



The gravitational force of attraction betⁿ the satellite & the earth will provide the necessary centripetal force

$$\frac{GMm}{(R+x)^2} = \frac{mv^2}{R+x}$$

$$\boxed{v = \sqrt{\frac{GM}{R+x}}}$$

$$\boxed{v = \sqrt{\frac{gR^2}{R+x}}}$$

When the satellite is orbiting very close to the surface of earth, $x \approx 0$

$$\boxed{v = \sqrt{\frac{GM}{R}} = \sqrt{gR}}$$

For earth, $v = 7.92 \ kms^{-1}$

Time period of satellite

Time taken by the satellite to go once around the earth.

$$T = \frac{\text{circumference of orbit}}{\text{Orbital velocity}}$$

$$= \frac{2\pi(R+x)}{v}$$

$$\boxed{T = 2\pi\sqrt{\frac{(R+x)^3}{GM}} = \frac{2\pi}{R}\sqrt{\frac{(R+x)^3}{g}}}$$

for $x \approx 0$, $T = 84.6$ minutes

Height of satellite

$$T^2 = \frac{4\pi^2 (R+x)^3}{R^2 g}$$

$$(R+x)^3 = \frac{T^2 R^2 g}{4\pi^2}$$

$$R+x = \left(\frac{T^2 R^2 g}{4\pi^2}\right)^{1/3}$$

$$x = \left(\frac{T^2 R^2 g}{4\pi^2}\right)^{1/3} - R$$

Energy of orbiting satellite

The potential energy of the satellite in the orbit at a height x is

$$U = -\frac{GMm}{R+x}$$

Since the satellite revolves in its orbit around the planet under the effect of its gravitational pull

$$\frac{mv^2}{R+x} = \frac{GMm}{(R+x)^2}$$

$$mv^2 = \frac{GMm}{R+x}$$

So, $T = \frac{1}{2} mv^2 = \frac{GMm}{2(R+x)}$

So, the total energy is

$$E = U + T$$

$$= -\frac{GMm}{R+x} + \frac{GMm}{2(R+x)}$$

$$= -\frac{GMm}{2(R+x)}$$

<u>Binding energy of an orbiting satellite</u>

$$B.E = -E = -\left(-\frac{GMm}{2(R+x)}\right)$$

$$B.E. = \frac{GMm}{2(R+x)}$$